UNIVERSITY OF MASSACHUSETTS
Department of Mathematics and Statistics
Basic Exam - Probability
Tuesday, January 18, 2022

Work all problems. 60 points are needed to pass at the Masters Level and 75 to pass at the Ph.D. level.

1. Suppose that a system has n parts. Let X_i be the lifetime of the i-th part of the system where $i = 1, \ldots, n$. Suppose that $X_1, \ldots, X_n$ are independent and that X_i has an exponential distribution with mean θ hours:

$$f(x_i \mid \theta) = \frac{1}{\theta} \exp\left(-\frac{x_i}{\theta}\right),$$

where $x_i > 0$, $\theta > 0$, $E(X_i) = \theta$, $Var(X_i) = \theta^2$, and the moment generating function of X_i is $M_{X_i}(t) = (1 - \theta t)^{-1}$ for $t < 1/\theta$.

   (a) Let $Y = \sum_{i=1}^{n} X_i$ be the total lifetime of the n parts. Find the **exact** distribution of Y.

   (b) Now, suppose we have 81 parts (n=81) and $\theta = 18$. **Approximate** the probability that the average lifetime of the 81 parts is between 14 and 16 hours. You can leave the final answer in terms of an integral.

   (c) The system works only if all n parts work. Let W be the lifetime of the system: W= the minimum of $X_1, \ldots, X_n$. Find the **exact** distribution of W.

   (d) Find the expected lifetime of the system using the result in (c).

2. Let X denote the number of claims in a fixed period of time from an insured in a pool of insureds. We assume that X has a Poisson distribution with mean $\theta > 0$. Some insureds are good risks (with small θ) and some are poor risks (with large θ). In order to reflect the risk characteristic of the insured, we regard θ as a random variable whose distribution is a Gamma distribution with shape parameter $\alpha > 0$ and scale parameter $\beta > 0$. That is, the conditional distribution of X conditional on θ and the unconditional distribution of θ are

$$f(x \mid \theta) = \exp(-\theta)\frac{\theta^x}{x!},$$
$$g(\theta) = \frac{1}{\Gamma(\alpha)\beta^\alpha}\theta^{\alpha-1} \exp\left(-\frac{\theta}{\beta}\right),$$

where $x = 0, 1, \ldots,$ $E(\theta) = \alpha\beta$ and $Var(\theta) = \alpha\beta^2$.

   (a) Compute the unconditional mean number of claims, $E(X)$.

(b) Compute the unconditional variance number of claims, $Var(X)$.

(c) Find the unconditional distribution of X.

3. $Y_1, \ldots, Y_n$ denote a random sample of size n from a distribution with probability density function

$$f(y) = \frac{2y}{\theta^2},$$

for $0 < y \le \theta < \infty$ and $f(y) = 0$, otherwise. Note that $E(Y_i^k) = 2\theta^k/(k+2)$ for $k = 1, 2, \ldots$ and $i = 1, \ldots, n$.
We consider a statistic $\bar{Y}_n = \frac{1}{n}\sum_{i=1}^n Y_i$.

(a) Show that $\bar{Y}_n$ converges in probability to $\frac{2\theta}{3}$.

(b) Find the **limiting** distribution of $\sqrt{n}\left(\bar{Y}_n - \frac{2\theta}{3}\right)$. Does the variance of this limiting distribution depend on θ?

(c) Find the **limiting** distribution of $\sqrt{n}\left(\log\left(\bar{Y}_n\right) - \log\left(\frac{2\theta}{3}\right)\right)$. Does the variance of this limiting distribution depend on θ?

(d) Using the result in (c), find $W_1(\bar{Y}_n, n)$ and $W_2(\bar{Y}_n, n)$, two statistics as a function of $\bar{Y}_n$ and n, such that $P[W_1(\bar{Y}_n, n) \le \theta \le W_2(\bar{Y}_n, n)]$ is **approximately** 0.99 [Hint: $P(Z > 1.644)$=0.05, $P(Z > 1.96)$=0.025 and $P(Z > 2.576)$=0.005 where $Z \sim N(0, 1)$].

4. Let X and Y be jointly normal random variables with finite means $E(X)$ and $E(Y)$, finite variances $Var(X)$ and $Var(Y)$, and finite covariance $Cov(X, Y)$.

(a) Consider the random variable $Y - \frac{Cov(X,Y)}{Var(X)}X$. Note that $\frac{Cov(X,Y)}{Var(X)}$ is constant and not random.
Show that the random variable $Y - \frac{Cov(X,Y)}{Var(X)}X$ follows a normal distribution and is independent of X.

(b) Using the result in (a), justify each of the steps in the following:

$$E(Y|X = x) = E\left(Y - \frac{Cov(X,Y)}{Var(X)}X + \frac{Cov(X,Y)}{Var(X)}X \,\middle|\, X = x\right) \tag{1}$$

$$= E\left(Y - \frac{Cov(X,Y)}{Var(X)}X\right) + \frac{Cov(X,Y)}{Var(X)}x \tag{2}$$

$$= E(Y) + \frac{Cov(X,Y)}{Var(X)}(x - E(X)). \tag{3}$$

(c) Now, assume that U and V are independent standard normal random variables. Let W_1 and W_2 be random variables defined by $W_1 = U - 3V + 2$ and $W_2 = 2U - 5V - 1$, respectively. What is the joint distribution of (W_1, W_2)?

(d) Use the results in (b) and (c) to compute $E(W_1 \mid W_2 = a)$ in terms of a where $W_1 = U - 3V + 2$ and $W_2 = 2U - 5V - 1$.