DEPARTMENT OF MATHEMATICS AND STATISTICS
UNIVERSITY OF MASSACHUSETTS AMHERST
APPLIED MATHEMATICS EXAM
JANUARY 2022

Do all six problems. All problems carry equal weight.
Passing level: 65% with at least three substantially correct.

1. In tests for car fuel economy, the vehicles are driven at constant speed V on a level highway. With no acceleration, the force of propulsion F must be in equilibrium with other forces, e.g., the air resistance. Assume the relevant variables are the velocity V, the rate C that fuel is burned (in volume per time), and the amount of energy K in a gallon of fuel (in mass per length per time-squared). Determine F as a function of V, C, K.

2. A simple fishery model is given as
$$\dot{N} = rN\left(1 - \frac{N}{K}\right) - H$$
where N is the fish population and H models the fishing effects. Show that a bifurcation occurs at a certain value H_c and classify the bifurcation. Discuss the long term behavior of the fish population for $H < H_c$ and $H > H_c$.

3. Consider the ODE system
$$\begin{aligned} \dot{x} &= -y + x(1 - x^2 - y^2 - x^4), \\ \dot{y} &= \quad x + y(1 - x^2 - y^2). \end{aligned}$$
Show that this system contains a stable limit cycle.

4. We consider the hyperbolic conservation law
$$u_t + ((1-u)u)_x = 0.$$
This can be used to model traffic density u where the cars have speed $1 - u$, moving slower in heavy traffic. We consider the initial condition
$$u(x,0) = \begin{cases} 1/4 & x \le 1/4 \\ x & 1/4 \le x \le 1 \\ 1 & 1 \le x \end{cases}.$$
Solve the equation using the method of characteristics. A shock will form. Resolve the motion of the shock.

5. Let $\varepsilon > 0$ be a small parameter. Find the leading order uniform approximation to the boundary value problem
$$\varepsilon y'' + (1 + x^2)y' + xy = 0, \qquad y(0) = A, y(1) = B.$$

6. An ant is doing a random walk on a one-dimensional lattice, with spacing Δx between the points. For each time step of length Δt, the ant either stays put with probability $\frac{1}{2}$ or moves left for **one lattice spacing** with probability p, or moves right for **two lattice spacings** with probability $(\frac{1}{2} - p)$. Assume it starts at $x = 0$ when time $t = 0$. Le $u(X_i, t_n)$ denote the probability that the ant is at $X_i = i\,\Delta x$ for $t = t_n$. Find the recurrence relationship between $u(X_i, t_{n+1})$ and those at time t_n. Find p so that the continuum limit (i.e., $\Delta t, \Delta x \to 0$) of the probability u is a pure diffusion equation, and find this equation when $D = \frac{\Delta x^2}{\Delta t}$ fixed.