Analysis Qualifying Examination

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This exam consists of eight equally weighted problems (ten points each): a passing grade is 65% (52/80), including at least five "essentially correct" problems ($\approx 7.5/10$).

Clearly show your work, explicitly stating or naming results that you use; justify use of named theorems by verifying necessary conditions.

Please work legibly and clearly label each page/file of your exam with your name.

1. Prove the following variant of Egoroff's Theorem for an arbitrary measure space $(X, \mathcal{M}, \mu)$: Suppose $g, f, f_1, f_2, \ldots$ are measurable functions on X with $f_n \to f$ almost everywhere, $|f_n| \le g$ for all n, and $g \in L^1(X)$. Then for every $\epsilon > 0$ there is $E \in \mathcal{M}$ such that $\mu(E) < \epsilon$ and $f_n \to f$ uniformly on E^c (the complement of E).
   Hint: Prove that $\lim_{n\to\infty} E_n(k) = 0$ for each positive integer k, where $E_n(k) = \cup_{m=n}^{\infty}\{|f_m - f| \ge k^{-1}\}$.

2. You may assume the conclusion of part (a) in proving part (b) (you don't have to).

   (a) Suppose $\mathcal{B}$ is a Banach space, $\mathcal{S}$ is a closed proper linear subspace (that is $\mathcal{S} \ne 0$ and $\mathcal{S} \ne \mathcal{B}$), and $f_0 \notin \mathcal{S}$. Show that there is a continuous linear functional $\ell : \mathcal{B} \to \mathbb{R}$ such that $\ell(f) = 0$ for $f \in \mathcal{S}$, $\ell(f_0) = 1$, and $\|\ell\| = 1/d$, where d is the distance from f_0 to $\mathcal{S}$.

   (b) Prove that a linear functional $\ell : \mathcal{B} \to \mathbb{R}$ is continuous if and only if $\{f \in \mathcal{B} \text{ s.t. } \ell(f) = 0\}$ is closed.

3. The underlying measure space in this problem is $\mathbb{R}^d$ with the Lebesgue measure m. Recall that the maximal function (the sup is over all balls

containing x),

$$f^*(x) = \sup_{x \in B} \frac{1}{m(B)} \int_B |f(y)| dy,$$

satisfies the estimate $m(\{|f^*| > \alpha\}) \leq A\alpha^{-1} \|f\|_{L^1}$ for all $\alpha > 0$ and all $f \in L^1$. Here A is a constant which is independent of α and f.

(a) Prove that there is a constant C (independent of α and f) such that for $f \in L^p \cap L^1$, $p \in (1, \infty)$,

$$m(\{f^* > \alpha\}) \leq \frac{C}{\alpha} \int_{\{|f| > \alpha/2\}} |f| dx.$$

Hint: Write $f = f_1 + f_2$ where $f_1 = \chi_{\{|f| > \alpha/2\}} f$ and $f_2 = \chi_{\{|f| \leq \alpha/2\}} f$.

(b) Prove that there is a constant M (which is independent of f) such that $\|f^*\|_{L^p} \leq M \|f\|_{L^p}$ for all $f \in L^p \cap L^1$, $p \in (1, \infty)$.
Hint: Recall that for any non-negative measurable function F, we have $\int_{\mathbb{R}^d} (F(x))^p dx = \int_0^\infty \lambda(\alpha^{1/p}) d\alpha$, where $\lambda(\alpha) = m(\{|F| > \alpha\})$. You may use this fact and the previous part.

4. Let $H = L^2(\mathbb{R}^d)$ with the Lebesgue measure, and let $B : H \times H \to \mathbb{C}$ be sesquilinear (linear in the first component and conjugate linear in the second), and satisfy

$$|B(f, g)| \leq C \|f\| \|g\| \tag{1}$$

for some constant C. Recall that as a consequence of the Riesz Representation Theorem, there is a unique bounded linear operator $T : H \to H$ such that $B(f, g) = \langle Tf, g \rangle$ for all $f, g \in H$. Let $B : H \to H$ be given by

$$B(f, g) = \int_{\mathbb{R}^d} \Big(\int_{\mathbb{R}^d} \frac{\widehat{f}(\xi)}{1 + |\xi|^2} e^{ix \cdot \xi} d\xi \Big) \Big(\int_{\mathbb{R}^d} \overline{\widehat{g}(\eta)} \sin(|\eta|) e^{-ix \cdot \eta} d\eta \Big) dx,$$

where $\widehat{f}(\xi) = \int_{\mathbb{R}^d} e^{-ix \cdot \xi} f(x) dx$ denotes the Fourier transform. Prove that B satisfies the estimate (1) (for some C independent of f, g) and give T as above in terms of Fourier transforms and their inverses (you're allowed to be off by factors of 2π).

5. Consider the locally integrable function $f : \mathbb{R}^2 \to \mathbb{R}$, given in polar coordinates as $f(r, \theta) = \log r$, where log is the natural logarithm, defined on $(0, \infty)$. Calculate the derivative $(\partial_r^2 + \frac{1}{r}\partial_r)f$ in the distributional sense.
Hint: Consider the integral in polar coordinates over the regions $\{0 \le r < \epsilon\}$ and $\{r \ge \epsilon\}$ separately and use integration by parts where needed.

6. In this problem each Euclidean space is equipped with the usual Lebesgue measure. Suppose $K : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ satisfies $\int_{\mathbb{R}^m} |K(x, y)|dy \le C$ for almost every x and $\int_{\mathbb{R}^n} |K(x, y)|dx \le C$ for almost every y, for some finite constant C. If $p \in (1, \infty)$ prove that

$$TF(x) = \int_{\mathbb{R}^m} K(x, y)f(y)dy$$

defines a bounded linear operator $T : L^p(\mathbb{R}^m) \to L^p(\mathbb{R}^n)$ with $\|T\| \le C$.

7. The two parts of this problem are unrelated.

   (a) Prove that there is a constant $C > 0$ such that for all Schwartz functions $f : \mathbb{R}^d \to \mathbb{R}$,

   $$\|x_j f\|_{L^2} \|\xi_j \widehat{f}\|_{L^2} \ge C\|f\|_{L^2}^2.$$

   Here x_j and ξ_j are the jth coordinate function on the spatial and Fourier domain, and $\widehat{\cdot}$ denotes the Fourier transform.
   Hint: Start with $\|f\|_{L^2}^2$ and integrate by parts.

   (b) We say that a subspace $S \subseteq L^2(\mathbb{R}^d)$ is *total* if its orthogonal complement $S^\perp$ satisfies $S^\perp = \{0\}$. For $f \in L^2(\mathbb{R}^d)$ prove that $S = \{f(x + a) \mid a \in \mathbb{R}^d\}$ is total if and only if $\widehat{f}(\xi) \ne 0$ a.e. (that is, $m(\{\widehat{f} = 0\}) = 0$).
   Hint: Convolutions.

8. Let H be a Hilbert space and $T : H \to H$ an isometry, that is, a bounded linear operator with $\|Tf\| = \|f\|$ for all $f \in H$. We will denote the adjoint of T by T^* and the identity map by I. You may use the conclusion of the first part in proving the second part of this problem.

(a) Let $S = \{f \in H \mid T(f) = f\}$, $S_* = \{f \in H \mid T^*(f) = f\}$, and $S_1 = \{f \in H \mid f = (I - T)g \text{ for some } g \in H\}$. Prove that $S = S_*$ and $(\overline{S_1})^\perp = S$.
Hint: For the first statement in one direction use the fact that for an isometry $T^*T = I$ and for the other consider $\langle f, (I - T^*)f \rangle$.

(b) Let $A_n = \frac{1}{n}(I + T + \cdots + T^{n-1})$. Prove that for each $f \in H$ we have

$$\lim_{n\to\infty} \|A_n(f) - P(f)\| = 0,$$

where P denotes the orthogonal projection on S (it is easy to see that S is closed, and you do not need to prove this).
Hint: Decompose $f = f_0 + f_1$ with $f_0 \in S$ and $f_1 \in \overline{S_1}$, and write $f_1 = (f_1 - f_2) + f_2$ where $f_2 \in S_1$ is very close to f_1. Then consider A_n on each term of the decomposition separately.