Basic Exam: Advanced Calculus & Linear Algebra

Department of Mathematics and Statistics
University of Massachusetts, Amherst
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Instructions: Do all the problems and show your work. The passing standards are:

- Master's level: 60% with three questions essentially, complete (including one question from each part);
- Ph.D. level: 75% with two questions from each part essentially complete.

Calculus

1. Evaluate $\int \frac{\ln(1+x)}{x^{3/2}} dx$. (*Hint:* At some point it will be useful that the derivative of $\arctan(t)$ is $1/(1+t^2)$.)
2. Find the global maxima and minima of the function $f\colon \mathbb{R}^3 \to \mathbb{R}$ given by
$$f(x, y, z) = 5x + y - 3z$$
on the region
$$X := \{(x, y, z) \in \mathbb{R}^3 \mid x + y + z = 0, x^2 + y^2 + z^2 = 1\}.$$
3. Consider the integral
$$\int_{-3}^{3} \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \int_{0}^{9-x^2-y^2} x^2 \, dz \, dy \, dx.$$
Draw a picture of its domain of integration and then evaluate the integral. (*Hint:* Try using other coordinate systems.)
4. Let E_r denote the ellipse given by solutions to the equation $r^2x^2 + y^2 = r^4$. For $r > 0$, let
$$f(r) = \int_{E_r} \frac{-y}{x^2 + y^2} dx + \frac{x}{x^2 + y^2} dy.$$
   a) State Green's theorem and use it to compute $f(r) - f(1)$ for any $r > 1$.
   b) Compute $f(1)$.
5. Let α be a real number that is not an integer. Associated to it is a sequence of numbers C_k^α where $C_0^\alpha = 1$, $C_1^\alpha = \alpha$, and
$$C_k^\alpha = \frac{\alpha(\alpha - 1) \cdots (\alpha - k + 1)}{k!}$$
for k a natural number. Consider the power series
$$F_\alpha(x) = \sum_{k=0}^{\infty} C_k^\alpha x^k.$$
   a) Find the radius of convergence for F_α.
   b) Show that F_α satisfies the differential equation
$$(1 + x)F_\alpha' = \alpha F_\alpha.$$
   c) Show that $G_\alpha(x) = (1 + x)^\alpha$ also solves this differential equation. (Make sure to discuss how to properly define G_α as a function on the real line. What is exponentiation with a real number?) How are F_α and G_α related?

Linear Algebra

6. Let
$$A = \begin{bmatrix} -2 & 4 & -2 & 4 \\ 2 & -6 & -3 & 1 \\ -3 & 8 & 2 & 1 \end{bmatrix}.$$

   a) If possible, find a solution for $A\mathbf{x} = \mathbf{e_1} - \mathbf{e_2}$. If not, explain why.
   b) Find a basis for $Nul(A)$ and $Row(A)$, motivating your choice.
   c) State the *rank theorem* relating the dimensions of $Col(A)$ and $Nul(A)$ and verify that it holds for the matrix A.

7. Sulphur-crested cockatoos migrate each month between Adelaide, Brisbane, and Canberra. On average, their migration pattern is:

   - 50% of the cockatoos in Adelaide remain in Adelaide, while 25% each go to Brisbane and Canberra.
   - 50% of the cockatoos in Brisbane remain in Brisbane, and the rest go to Adelaide.
   - 50% of the cockatoos in Canberra remain in Canberra, and the rest go to Adelaide.

   a) If we write the populations in each city by alphabetical order, we get a column vector P of height 3. Write the matrix T that TP is the expected population distribution next month.
   b) Find the expected distribution of the sulphur-crested cockatoos population at month k, i.e., describe T^kP.
   c) As $k \to \infty$, an equilibrium is reached. Describe the relative proportions between the three populations in this limit (i.e., the long time behavior of this dynamical system).

8. Let T be the linear transformation with standard matrix given by
$$\begin{bmatrix} 0 & 0 & a & 1 \\ 1 & 1 & -8 & 4 \\ 0 & 0 & b & c \\ 0 & 3 & -2 & -1 \end{bmatrix}.$$

   a) What is the domain of T? And the codomain?
   b) For which values of a and b is T one-to-one?
   c) For which values of a and b is T onto?
   d) For which values of a and b is T invertible?

9. Let E be the ellipse whose equation is given by
$$4x_1^2 + 4x_1x_2 + 2x_2^2 = 1.$$

   a) Find a linear transformation $T\colon \mathbb{R}^2 \to \mathbb{R}^2$ such that $T(E)$ is the circle in $\mathbb{R}^2$ centered at the origin and with radius $\frac{1}{2}$. (*Hint:* It may be useful to rewrite $2x_2^2$ in an equivalent form.)
   b) Find the area of the region in $\mathbb{R}^2$ bounded by the ellipse E.

10. Say if each of the following is true or false, giving justifications or counterexamples as appropriate.

    a) If $\mathbf{v}$ is an eigenvector for an $n \times n$-matrix A and for an $n \times n$-matrix B, then $\mathbf{v}$ is also an eigenvector for the matrix $BA + I_n$.
    b) There exists a matrix M with real coefficients such that
$$M^4 = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}.$$

    c) If a nonzero vector $\mathbf{v}$ belongs to the orthogonal complement of $Span\{\mathbf{w}\}$ and $\mathbf{w}$ is a nonzero vector, then $\mathbf{w}$ cannot be a scalar multiple of $\mathbf{v}$.