ADVANCED EXAM: TOPOLOGY, WINTER 2021

Answer all seven questions. Justify your answers. Passing standard: 70% **with four questions essentially complete.**

Problem 1 Consider the following topologies on the real line $\mathbb{R}$:
(i) trivial topology, (ii) discrete topology, (iii) finite complement topology.
(Recall that *finite complement topology* is the topology generated by a basis consisting of complements of finite subset.)

For each topology, determine, with explanations, which one of the following functions from $\mathbb{R} \to \mathbb{R}$ (both the domain and the range taken with the same topology)

$$f(x) = x^4 \ , \quad g(x) = e^x \ , \quad h(x) = cos(x)$$

are **(a)** continuous, **(b)** open maps, **(c)** embeddings.

Problem 2 Prove that a metric space has a countable dense subset if and only if it has a countable basis for its topology.

Problem 3. Let M_i be n–dimensional manifolds, for $i = 1, 2$. Let $M = M_1 \# M_2$ be their connected sum. (This is obtained by taking out regular coordinate balls B_i from M_i and then identifying $\partial(M_1 \setminus B_1)$ with $\partial(M_2 \setminus B_2)$ via some homeomorphism.) Show that: (a) M is also an n–manifold. (b) If M_1 and M_2 are connected and $n > 1$, then so is M. (c) If M_1 and M_2 are compact, then so is M.

Problem 4. Classify all connected coverings of $\mathbb{RP}^2 \vee S^1$. Tell which ones are regular.

Problem 5. Compute the fundamental group of the space obtained from the 2–torus $T^2 = S^1 \times S^1$ and the Möbius band Mb by identifying the circle $S^1 \times \{x_0\}$ on T^2 (where (x_0 is a fixed point on S^1) with the boundary ∂Mb.

Problem 6. Let X be the quotient of S^3 qiven by the equivalence relation $x \sim -x$ for all x on the equator $x \in S^2 \subset S^3$. Compute the homology groups $H_i(X; \mathbb{Z})$ and $H_i(X; \mathbb{Z}_2)$.

Problem 7. Prove that a compact manifold with boundary does not retract onto its boundary.