DEPARTMENT OF MATHEMATICS AND STATISTICS
UNIVERSITY OF MASSACHUSETTS AMHERST
APPLIED MATHEMATICS EXAM
JANUARY 2021

Do all six problems. All problems carry equal weight.
Passing level: 65% with at least three substantially correct.

1 A physical system is described by a unit-free law $f(E, P, A) = 0$, where E, P, A are energy, pressure, and area, respectively. Show that $PA^{3/2}E^{-1} = \text{const}$.

2 Show that the system

$$x' = y + \frac{x(1 - x^2 - y^2)}{\sqrt{x^2 + y^2}}$$
$$y' = -x + \frac{y(1 - x^2 - y^2)}{\sqrt{x^2 + y^2}}$$

has a stable limit cycle. Find the ω-limit set of $x_0 = (2019, 2020)$.

3 Consider equation

$$x' = -x(x^2 - 2x - \mu)\,,$$

where $\mu \in \mathbb{R}$ is a parameter.

(a) Find all equilibria as functions of μ and determine their stability.

(b) Sketch the bifurcation diagram. Find and classify the bifurcation point.

4 Let X_n denote a random walk on a one dimensional lattice, with spacing Δx between the points. For each time step of length Δt the process does one of the following:

- step two spaces left (distance $2\Delta x$) with probability $\frac{1}{2}$,
- step one space right with probability p,
- step three spaces right with probability $\frac{1}{2} - p$.

Find p so that the resulting equation is a pure diffusion. Compute the continuum limit of the probability density as $\Delta x \to 0, \Delta t \to 0$ with $D = \frac{\Delta x^2}{\Delta t}$ fixed.

5 Find the two leading order terms in the approximation for the four roots of

$$\epsilon x^4 - x^2 + 3x - 2 = 0$$

in terms of the small positive parameter ϵ.

6 Find and classify the equilibria for the system:

$$x' = (\sin x + x)e^y,$$
$$y' = (\cos^2 x)(y^2 - 1).$$

Give both the type and stability for each point.