Analysis Qualifying Examination

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Friday, January 29th, 2021

This exam consists of eight equally weighted problems (ten points each): a passing grade is 65% (52/80), including at least five "essentially correct" problems ($\approx 7.5/10$).

Clearly show your work, explicitly stating or naming results that you use; justify use of named theorems by verifying necessary conditions.

Please work legibly and clearly label each page/file of your exam with your name.

1. Let E be a measurable set in $\mathbb{R}^d$ and let m denote Lebesgue measure on $\mathbb{R}^d$. Define $E_n := \{x \in \mathbb{R}^d \mid \; d(x, E) < \frac{1}{n}\}$, where $d(x, E)$ denotes the distance from x to E.

   (a) If E is compact show that $m(E) = \lim_{n\to\infty} m(E_n)$.

   (b) Is the same statement true if E is closed, but not bounded? Prove or give a counter example.

   (c) Is the same statement true if E is bounded, but not closed? Prove or give a counter example.

2. Let $C \subset [0, 1]$ be a generalized Cantor set of measure $\eta > 0$, denote $G = C^c$, and define

$$g(x) = \int_0^x \chi_G(t)\, dt.$$

   Show that g is absolutely continuous and strictly monotone, but satisfies $g' = 0$ on a set of positive measure.

3. Recall that Hardy's maximal function of f is given by

$$f^*(x) := \sup_{x\in B} \frac{1}{m(B)} \int_B |f(y)|\, dy.$$

Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \frac{1}{x\,(\log x)^2} \quad \text{for} \quad 0 < x < 1/e,$$

and $f(x) = 0$ otherwise. Check that f is integrable, and calculate its integral $F(x) := \int_{(-\infty,x)} f(t)\,dt$. However, show that f^* is *not* locally integrable by computing $\int_{(0,r)} f^*(x)\,dx = \infty$ for $r > 0$.
[You may use calculus techniques to integrate without further justification.]

4. Recall that for smooth and compactly supported $f : \mathbb{R}^d \to \mathbb{R}$,

$$\|\nabla f\|_{L^r}^r := \int_{\mathbb{R}^d} |\nabla f|^r dx = \int_{\mathbb{R}^d} \Big(\sum_{i=1}^d \big(\frac{\partial f}{\partial x^i}\big)^2 \Big)^{\frac{r}{2}} dx,$$

and denote $\partial^\alpha f := (\frac{\partial}{\partial x^1})^{\alpha_1} \dots (\frac{\partial}{\partial x^n})^{\alpha_n} f$ for multi-index α, and $|\alpha| = \sum \alpha_i$.

(a) Prove that the following statement is false: There exists a constant C such that for all smooth compactly supported $f : \mathbb{R}^d \to \mathbb{R}$,

$$\|f\|_{L^\infty} \le C \|f\|_{L^p} \|\nabla f\|_{L^r}, \qquad \frac{d}{p} + \frac{d}{r} \ne 1.$$

Hint: Consider a nonzero f and study the inequality for $f_\lambda(x) = f(\lambda x)$.

(b) Assume without proof that for some $k > \frac{d}{2}$, there is a constant C such that for all smooth compactly supported $f : \mathbb{R}^d \to \mathbb{R}$

$$\|f\|_{L^\infty} \le C (\|f\|_{L^2} + \sum_{|\alpha|=k} \|\partial^\alpha f\|_{L^2}).$$

Prove that there is a constant $\tilde{C}$ such that for all f as above,

$$\|f\|_{L^\infty} \le \tilde{C} \|f\|_{L^2}^{\frac{2k-d}{2k}} \Big(\sum_{|\alpha|=k} \|\partial^\alpha f\|_{L^2} \Big)^{\frac{d}{2k}}.$$

Hint: Consider $f_\lambda(x) = f(\lambda x)$ for appropriate λ.

5. Show that a normed vector space X is a Banach space if and only if the unit sphere $\{x \in X \mid \|x\| = 1\}$ is complete.

6. Let X and Y be two nonempty normed spaces, and let $L(X, Y)$ denote the space of linear maps from X to Y with the usual norm $\|T\| = \sup_{x \in X} \frac{\|Tx\|_Y}{\|x\|_X}$. Prove that if $L(X, Y)$ is a Banach space, then Y is also a Banach space.
[Hint: Given a sequence $\{y_n\} \subset Y$ and $f \in X^*$, consider the maps T_n given by $T_n(x) = f(x)y_n$.]

7. Show that any integrable function $f : \mathbb{R} \to \mathbb{R}$ which is polynomially bounded (that is, there is a polynomial $p(x)$ such that $|f(x)| \leq p(x)$ for all x) defines a tempered distribution. Does e^{ax} define a tempered distribution? Why or why not?

8. For $k \in \{0, 1, 2, 3, ...\}$ let μ_k be the measure on $\mathbb{Z}$ define by $\mu_k(n) = (1 + n^2)^k$. Let $H_k = L^2(\mathbb{Z}, \mu_k)$. Here we identify functions on $\mathbb{Z}$ with two sided sequences $x = (\dots, x_{-1}, x_0, x_1, \dots)$, with $x_n \in \mathbb{R}$. We also denote the $L^2(\mathbb{Z}, \mu_k)$ inner product by $\langle \cdot, \cdot \rangle$, that is

$$\langle x, y \rangle = \sum_{n \in \mathbb{Z}} (1 + n^2)^k x_n y_n.$$

Then H_k is a Hilbert space and $H_{k_2} \subseteq H_{k_1}$ whenever $k_1 < k_2$ (you may assume these assertions which are easy to verify).

(a) Prove that finite sequences (i.e., sequences x as above such that $x_n = 0$ for all but finitely many n) are dense in H^k for all $k \geq 0$.

(b) Suppose $k_2 > k_1$. Show that the unit ball in H_{k_2} is relatively compact in H_{k_1} (that is, the closure of the unit ball of H_{k_2} is compact in H_{k_1}).