DEPARTMENT OF MATHEMATICS AND STATISTICS
UNIVERSITY OF MASSACHUSETTS, AMHERST

ALGEBRA EXAMINATION

JANUARY 2021

Passing Standard: The passing standard is 70% with essentially correct solutions to five problems. Show all your work and justify your answers carefully. All rings contain the identity and all ring homorphisms preserve the identity.

1. Group theory

1. Construct a non-abelian group of order 75.

2. Let G be the group $\mathbf{Z}/p\mathbf{Z} \times \mathbf{Z}/p^2\mathbf{Z} \times \mathbf{Z}/p^3\mathbf{Z}$, where p is a prime.

(1) Determine the number of cyclic subgroups of G of order p^3.
(2) Determine the number of subgroups (not necessarily cyclic) of G of order p^3.

2. Ring theory

3. Let $f : R \to S$ be a homomorphism of commutative rings.

(1) Show that if P is a prime ideal of S, then its preimage $f^{-1}(P)$ is a prime ideal of R.
(2) Show that if f is surjective and M is a maximal ideal of S, then its preimage $f^{-1}(M)$ is a maximal ideal of R.
(3) Give an example of a non-surjective $f : R \to S$ and a maximal ideal M of S such that $f^{-1}(M)$ is not a maximal ideal of R.

4.

(1) Let R be an integral domain, and let $I \subset R$ be a principal ideal. Prove that the R-module $I \otimes_R I$ has no torsion elements.
(2) What if R is not an integral domain? Either prove the statement or give a counterexample.

5. Let A be a 2×2 matrix with entires in $\mathbf{Q}$. Assume that $A^3 = I$, the 2×2 identity matrix, yet $A \neq I$.

(1) Find the rational canonical form of A.
(2) Find the Jordan canonical form of A, thought of as a matrix over $\mathbf{C}$.

3. Field theory

6.

(1) Compute the minimal polynomial of $\alpha = \sqrt{2 + \sqrt{2}}$.
(2) Compute the Galois closure K of the extension $\mathbf{Q}(\alpha)/\mathbf{Q}$, and all subfields of K.

7. Let F denote a finite field of order 2^n for some $n \geq 1$. Determine all n such that the polynomial $x^2 + x + 1$ is irreducible in $F[x]$.

Reference

Let $f = x^4 + px^2 + qx + r$. The discriminant of f is

$$16p^4r - 4p^3q^2 - 128p^2r^2 + 144pq^2r - 27q^4 + 256r^3.$$

The resolvent cubic of f is

$$x^3 - 2px^2 + (p^2 - 4r)x + q^2.$$