UNIVERSITY OF MASSACHUSETTS
Department of Mathematics and Statistics
Basic Exam - Probability
Friday, January 17, 2020

Work all problems. 60 points are needed to pass at the Masters Level and 75 to pass at the Ph.D. level.

1. Suppose a cell phone is missing and it is presumed that it was equally likely to have gone missing in any of 3 possible classrooms. Let θ_i denote the "overlook" probability that the cell phone is not found upon a search of classroom i given that it is actually in classroom i, for $i = 1, 2, 3$. Thus, $1 - \theta_i$ is the probability that the cell phone is found in classroom i upon a search of the classroom i, given that it is actually there.
What is the conditional probability that the cell phone is in classroom 1, given that the search of classroom 1 was unsuccessful?

2. The random vector $(Y, Z)^T$ follows a bivariate Normal distribution with mean vector $(0, 0)^T$ and covariance matrix $\begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}$, and its probability density function is

$$f(y, z) = \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left[-\frac{y^2 - 2\rho yz + z^2}{2(1-\rho^2)}\right].$$

   (a) Show that the conditional distribution of Y given Z is a normal distribution with mean ρZ and variance $1 - \rho^2$.
   (b) Describe the behavior of the conditional distribution of Y given Z (including its mean and variance) as $\mid \rho \mid$ approaches 1.
   (c) Define $U = Y + Z$ and $V = Y - Z$. Obtain the marginal distributions of U and V, respectively.
   (d) Compute the covariance between U and V, $Cov(U, V)$. Are U and V independent? Justify your answer.

3. Suppose that X is a Bernoulli random variable from Bernoulli trial with the success probability θ, denoted as $X \sim Bernoulli(\theta)$, where $0 < \theta < 1$. A generalization of the Bernoulli distribution is to allow the success probability to vary from trial to trial, keeping the trials independent :

$$X_i \mid \Theta_i \sim Bernoulli(\Theta_i),\ i = 1, \ldots, n,$$
$$\Theta_i \sim Beta(\alpha, \beta)$$

where Θ_i is a Beta random variable with the probability density function

$$f(\Theta) = \frac{1}{B(\alpha, \beta)} \Theta^{\alpha-1}(1-\Theta)^{\beta-1},$$

$\alpha, \beta > 0$ and $B(\alpha, \beta)$ is a normalization constant to ensure that $f(\Theta)$ is the probability density function. Note that the mean and variance of Θ are $E(\Theta) = \frac{\alpha}{\alpha+\beta}$ and $Var(\Theta) = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$, respectively.

A random variable of our interest is the total number of successes, $Y = \sum_{i=1}^n X_i$.

(a) Compute the mean of Y, $E(Y)$.

(b) Compute the variance of Y, $Var(Y)$.

4. A real-valued random variable X is said to follow the Weibull distribution with scale $\lambda \in (0, \infty)$ and shape $k \in (0, \infty)$, denoted as Weibull(λ, k), if it has distribution function $F(x; \lambda, k)$ given by

$$P(X \leq x; \lambda, k) = F(x; \lambda, k) = 1 - e^{-(x/\lambda)^k}$$

for $x > 0$, and 0 otherwise, and density function $f(x; \lambda, k)$ given by

$$f(x; \lambda, k) = \frac{k}{\lambda}\left(\frac{x}{\lambda}\right)^{k-1} e^{-(x/\lambda)^k}$$

again for $x > 0$, and 0 otherwise.

Suppose that $X_1, X_2, \ldots$ is a sequence of IID Weibull(λ, k) random variables.

(a) Show that (i) $E\left[X_i^k\right] = \lambda^k$ and (ii) $\text{Var}(X_i^k) = \lambda^{2k}$. (Hint: what is the distribution of X_i^k?)

Suppose that k is known. Define the statistic $\hat{\lambda}_n := \left[\frac{1}{n}\sum_{i=1}^n X_i^k\right]^{1/k}$.

(b) Show that $\hat{\lambda}_n \xrightarrow{p} \lambda$. (Hint: first show that $\hat{\lambda}_n^k \xrightarrow{p} \lambda^k$.)

(c) Show that $n^{1/2}\left(\hat{\lambda}_n - \lambda\right) \xrightarrow{d} N(0, v)$, and find the constant v. (Hint: first show that $n^{1/2}\left(\hat{\lambda}_n^k - \lambda^k\right) \xrightarrow{d} N(0, v'))$.

(d) Find the distribution of $\min\{X_1, \ldots, X_n\}$.

5. Suppose that $X_1, X_2, \ldots$ is a sequence of IID random real-valued variables with mean μ and variance $\sigma^2 \in (0, \infty)$. Consider the t-statistic

$$T_n = \frac{\sqrt{n}\left(\overline{X}_n - \mu\right)}{S_n},$$

where $\overline{X}_n = \frac{1}{n}\sum_{i=1}^n X_i$ and $S_n = \frac{1}{n-1}\sum_{i=1}^n \left(X_i - \overline{X}_n\right)^2$.

(a) Show that $\sqrt{n}\left(\overline{X}_n - \mu\right)$ converges in distribution, and find its limit distribution.

(b) Noting that $S_n^2 = \frac{n}{n-1}\left[\frac{1}{n}\sum_{i=1}^n (X_i - \mu)^2 - \left(\overline{X}_n - \mu\right)^2\right]$, show that $S_n^2 \xrightarrow{p} \sigma^2$.

(c) Show that T_n converges in distribution to $N(0, 1)$.

(d) Let $t_{k,\alpha}$ be the $1 - \alpha$ quantile of the t distribution with k degrees of freedom, i.e. $P(t_k \geq t_{k,\alpha}) = \alpha$ for t_k a t-distributed random variable with k degrees of freedom. Assuming that $t_{k,\alpha} \to z_\alpha$ as $k \to \infty$, where z_α is the $1 - \alpha$ quantile of the $N(0, 1)$ distribution, show that $P\left(T_n \geq t_{n-1,\alpha}\right) \to \alpha$.