DEPARTMENT OF MATHEMATICS AND STATISTICS
UNIVERSITY OF MASSACHUSETTS, AMHERST

ALGEBRA EXAMINATION

JANUARY 2019

Passing Standard: To pass the exam it is sufficient to solve five problems including a least one problem from each of the three parts. Show all your work and justify your answers carefully. All rings contain identity and all ring homorphisms preserve the identity.

1. Group theory

1. Give an example of a 3-Sylow subgroup of the symmetric group S_9 and show that it is isomorphic to a semi-direct product of abelian groups.

2. Let G be a non-abelian group of order 28 containing an element of order 4. Describe G in terms of generators and relations.

3. Let G be a finite group and p a prime dividing $\#G$. Suppose H is a subgroup of G of index p.
(a) What are the possibilities for the number of conjugate subgroups of H?
(b) Suppose in addition that p is the smallest prime dividing $\#G$. Prove that H is normal in G.

2. Ring theory

4. Let R be a reduced (that is, R has no non-zero nilpotent elements) commutative nonzero ring that has a unique prime ideal. Show that R is a field.

5. Let

$$R = \{(a, b) \in \mathbf{Z} \times \mathbf{Z} \mid a \equiv b \pmod 5\}.$$

Determine, with proof, all ring homomorphisms $R \to \mathbf{C}$.

6. Let $R = \mathbf{C}[x, y]$ and $I = (x, y) \subseteq R$. Consider the R-module $I \otimes_R I$.
(a) Show that there is a homomorphism of R-modules

$$I \otimes_R I \to \mathbf{C}$$

defined on pure tensors by

$$a \otimes b \mapsto \frac{\partial a}{\partial x}(0,0) \cdot \frac{\partial b}{\partial y}(0,0).$$

Here we define the R-module structure on $\mathbf{C}$ by

$$a \cdot \lambda = a(0,0) \cdot \lambda$$

for $a \in R$ and $\lambda \in \mathbf{C}$.
(b) Show that $x \otimes y - y \otimes x$ is a non-zero torsion element of $I \otimes_R I$ with annihilator I.

3. Field theory

7. Let K be the splitting field of the polynomial $x^4 - 4$ over $\mathbf{Q}$. Determine the Galois group $\mathrm{Gal}(K/\mathbf{Q})$.

8. Let K be a finite field and let L be an extension of K of degree n. Fix a monic irreducible polynomial $f \in K[x]$ of degree d dividing n. Show that there is $\alpha \in L$ which has minimal polynomial f over K.

9. Let $K \subseteq L \subseteq M$ be a tower of field extensions such that L/K and M/L are Galois. Does it follow that M/K is Galois? Give a proof or a counterexample and justify your answer.