DEPARTMENT OF MATHEMATICS AND STATISTICS
UNIVERSITY OF MASSACHUSETTS
BASIC EXAM - PROBABILITY
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Work all problems. Show all work. Explain your answers. State the theorems used whenever possible. 60 points are needed to pass at the Masters level and 75 at the Ph.D. level. Each question is worth 20 points.

1) Suppose Y_1 is a binomial random variable with n_1 trials and probability of success p, and Y_2 is a binomial random variable with n_2 trials and probability of success p. Assume that Y_1 and Y_2 are independent. Note that the $Bin(n,p)$ probability mass function is:

$$\binom{n}{y} p^y (1-p)^{n-y}, \quad y=0,...,n, \quad 0 \le p \le 1.$$

a) Find the conditional distribution of Y_1 given that $Y_1 + Y_2 = k$. Give the probability mass function of this conditional distribution and identify it by its family name and parameters.

b) Find $\Pr(Y_1 > Y_2)$.

c) Find $\Pr(Y_1 / Y_2 = 1)$.

2) Let $X \sim N(\mu, \sigma^2)$

a) Show that the moment generating function of X is:

$$M_X(t) = \exp(\mu t + \sigma^2 t^2 / 2)$$

b) Show that if $X_1 \sim N(\mu_1, \sigma_1^2)$, $X_2 \sim N(\mu_2, \sigma_2^2)$, and X_1 and X_2 are independent, then $X_1 + X_2 \sim N(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$.

3) Suppose that an individual can be either healthy (H) or have a disease (D). A blood test can show that an individual is diseased (positive) or healthy (negative). In a specific population, the following probabilities are given:

-the probability an individual is H and the test is positive is 0.098
-the probability an individual is H and test is negative is 0.892
-the probability an individual is D and the test is positive is 0.008
-the probability an individual is D and the test is negative is 0.002

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a) Suppose an individual is randomly selected from this specific population. What is the probability that the individual is healthy?

b) Given that the individual is diseased, what is the probability that the test is positive?

c) Suppose that an individual takes the test and the test is positive. What is the probability that the individual has the disease?

4) Let $U_1, U_2, ..., U_n$ be i.i.d. Uniform[0,1] random variables. Let $U_{(n)} = \max_i U_i$.

a) Find the cumulative distribution function for $U_{(n)}$.

b) Show that $U_{(n)}$ converges to 1 in probability.

c) Show that $n(1 - U_{(n)})$ converges in distribution to the Exponential(1) distribution; recall that $X \sim$ Exponential(1) is equivalent to X having the probability density function

$$f_X(x) = \begin{cases} e^{-x} & \text{for } x \geq 0 \\ 0 & \text{else.} \end{cases}$$

5) Let $X_1, X_2, X_3, ...$ be i.i.d. random variables with common probability density function equal to

$$f_X(x) = \begin{cases} 1/2 & \text{if } x \in [-1,1] \\ 0 & \text{else.} \end{cases}$$

Let Z be a discrete random variable with $P(Z=1) = P(Z=-1) = P(Z=0) = 1/3$, and let Z be independent of $\{X_i\}_{i=1}^{\infty}$. For each $i = 1,2,3,...$, let $Y_i = X_i \cdot Z$.

a) Show that the Y_i's are uncorrelated but not independent.

b) Let $W_n = \dfrac{\sum_{i=1}^{n} X_i}{\sqrt{n}}$. What is the expectation of W_n? What is the variance of W_n?

c) Let $V_n = \dfrac{\sum_{i=1}^{n} Y_i}{\sqrt{n}}$. What is the expectation of V_n? What is the variance of V_n?