NAME:

Advanced Analysis Qualifying Examination
Department of Mathematics and Statistics
University of Massachusetts

Tuesday, January 16th, 2018

Instructions

1. This exam consists of eight (8) problems all counted equally for a total of 100%.

2. You are encouraged to try to solve every problem; there is no penalty for incorrect answers.

3. In order to pass this exam, it is enough that you solve essentially correctly at least five (5) problems and that you have an overall score of at least 65%.

4. State explicitly all the results that you use in your proofs and <u>verify</u> that these results apply.

5. Show all your work and <u>justify</u> the steps in your proofs.

6. Please write your full work and answers <u>clearly</u> in the blank space under each question and on the blank page after each question.

Conventions

1. If a measure is not specified, use Lebesgue measure on $\mathbb{R}^d$. This measure is denoted by m.

2. If a σ-algebra on $\mathbb{R}^d$ is not specified, use the Borel σ-algebra.

1. Let (X, μ) be a finite measure space with $\mu(X) = 1$. The two parts of this problem are unrelated.

(a) Given $f \in L^1(X)$ define $\lambda = \int_X f d\mu$ and $\sigma_m = (\int_X |f - \lambda|^m d\mu)^{1/m}$, $m = 1, 2, \ldots$. Prove that for any $k > 0$ and any $m = 1, 2, \ldots,$

$$\mu(\{x \in X \text{ s.t. } |f(x) - \lambda| \geq k\sigma_m\}) \leq k^{-m}$$

.

(b) Let $\{A_n\}_{n=1}^{\infty}$ be a sequence of measurable sets in X with $\sum_{n=1}^{\infty} \mu(A_n) < \infty$. Define $A = \cap_{k=1}^{\infty} \cup_{n=k}^{\infty} A_n$. Clearly A is measurable. Prove that $\mu(A) = 0$.

2. The three parts of this problem are unrelated.

(a) Recall the definition of convolution: $(f * g)(x) := \int_{\mathbb{R}^d} f(x-y)\,g(y)\,dy$. Suppose f and g are in $L^1(\mathbb{R}^d)$. You may assume that $f(x-y)g(y)$ is measurable on $\mathbb{R}^{2n}$ with the usual product measure. Show that $f * g \in L^1(\mathbb{R}^d)$ and that in fact $\|f * g\|_{L^1(\mathbb{R}^d)} \leq \|f\|_{L^1(\mathbb{R}^d)} \|g\|_{L^1(\mathbb{R}^d)}$.

(b) Show that if $f \in L^1(\mathbb{R}^d)$ and g is a bounded function defined on $\mathbb{R}^d$ then the convolution $f * g$ (defined as in part (a)) is a continuous function on $\mathbb{R}^d$. Hint: Use the definition of continuity.

(c) Suppose $f \in L^1(\mathbb{R}^d)$ and for any $\alpha \geq 0$ let $E_\alpha := \{x \in \mathbb{R}^n \text{ s.t. } |f(x)| \geq \alpha\}$. Show that

$$\|f\|_{L^1(\mathbb{R}^d)} = \int_0^\infty m(E_\alpha)\,d\alpha$$

where m denotes the Lebesgue measure on $\mathbb{R}^d$. Hint: Use Fubini-Tonelli.

3. Suppose the family of measurable functions $\{K_\delta\}_{\delta>0}$ satisfy the following three conditions:

$$(i) \quad \int_{\mathbb{R}^d} K_\delta(x)dx \;=\; 1,$$

$$(ii) \quad \int_{\mathbb{R}^d} |K_\delta(x)|\, dx \le M, \quad M \text{ independent of } \delta,$$

$$(iii) \quad \lim_{\delta\to 0} \int_{\{y\in\mathbb{R}^d \text{ s.t. } |y|\ge\eta\}} |K_\delta(x)|\, dx \;=\; 0, \quad \text{for any } \eta > 0.$$

Show that if f is a bounded measurable function on $\mathbb{R}^d$ which is continuous at some $x_0 \in \mathbb{R}^d$ then

$$\lim_{\delta\to 0} \int_{\mathbb{R}^d} f(x_0 - y)K_\delta(y)dy = f(x_0).$$

Hint: Note that $f(x) \;=\; \int_{\mathbb{R}^d} K_\delta(y)\, f(x)\, dy$ for any $x \in \mathbb{R}^d$ (why?).

4. Suppose $f \in L^p(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ for some $p > 1$.

(a) Show that $f \in L^q(\mathbb{R}^d)$ for any $q > p$. Moreover, show that $\{x \in \mathbb{R}^n \text{ s.t. } |f(x)| \geq a\}$ has finite Lebesgue measure for any $a > 0$.

(b) Show that $\lim_{q\to\infty} \|f\|_{L^q(\mathbb{R}^d)} = \|f\|_{L^\infty(\mathbb{R}^d)}$.

Hint: To show that $\lim_{q\to\infty} \|f\|_{L^q(\mathbb{R}^d)} \geq \|f\|_{L^\infty(\mathbb{R}^d)}$, for any $0 < \delta < \|f\|_{L^\infty(\mathbb{R}^d)}$ consider the L^q norm of f on the set $E_\delta := \{x \in \mathbb{R}^d \text{ s.t. } |f(x)| \geq \|f\|_{L^\infty(\mathbb{R}^d)} - \delta\}$.

5. (a) Show that if $\{f_n\}$ converges to f in $L^p(\mathbb{R}^d)$ for some $p \geq 1$ then f_n converges to f in measure, that is, show that for every $\varepsilon > 0$, $m(\{x \in \mathbb{R}^d : |f_n(x) - f(x)| > \varepsilon\}) \to 0$ as $n \to \infty$.

(b) Give an example of a sequence of functions on $\mathbb{R}$, with the Lebesgue measure, which converge to zero in measure but not in $L^1(\mathbb{R})$.

(c) Give an example of a sequence of functions on $\mathbb{R}$, with the Lebesgue measure, which converge to zero almost everywhere but not in measure.

(d) Give an example of a sequence of functions on $[0, 1]$, with the Lebesgue measure, which converge to zero almost everywhere and in measure, but not weakly in $L^p([0, 1])$ for any $p \geq 1$.

6. (a) Show that $f(x) = x^2 \cos^2(\pi/x)$ is a function of bounded variation on $[0, 1]$.

<u>Hint:</u> You may use a well-known theorem about functions of bounded variation.

(b) Show that $g(y) = \sqrt{y}$ is a function of bounded variation on $[0, 1]$.

<u>Hint:</u> You may use a well-known theorem about functions of bounded variation.

(c) If $f, g : [0, 1] \to [0, 1]$ are of bounded variation, is the composition $g \circ f$ always of bounded variation on $[0, 1]$?

<u>Hint:</u> Consider the examples from the previous two parts.

(d) Show that if $f : [0, 1] \to [0, 1]$ and $g : [0, 1] \to \mathbb{R}$ are such that g is Lipschitz and f is of bounded variation on $[0, 1]$, then $h = g \circ f$ is of bounded variation on $[0, 1]$.

<u>Hint:</u> Recall that g being Lipschitz on $[0, 1]$ means that there exists $L > 0$ such that $|g(x) - g(y)| \leq L|x - y|$ for all $x, y \in [0, 1]$.

7. Let $(X, \mathcal{M})$ be a measurable space.

(a) Let ν be a σ-finite measure on $(X, \mathcal{M})$. Assume that there exists a nonnegative measurable function g on X having the property that for all $A \in \mathcal{M}$,

$$\nu(A) = \int_A g \, d\nu.$$

Show that $g = 1$ almost everywhere with respect to ν.

(b) Let ρ and λ be σ-finite measures on $(X, \mathcal{M})$ having the property that $\rho \ll \lambda$ and $\lambda \ll \rho$ (i.e., each is absolutely continuous with respect to the other). Prove that the Radon-Nikodym derivatives satisfy

$$d\rho/d\lambda = \frac{1}{d\lambda/d\rho} \quad \text{a.e. with respect to either } \rho \text{ or } \lambda.$$

8. A family of functions $\{f_n\}_{n=1}^{\infty}$ on $\mathbb{R}^d$, with the Lebesgue measure m, is said to be *equi-integrable* if for any $\epsilon > 0$ there is $\delta > 0$ such that for any measurable set B with $m(B) < \delta$, then

$$\int_B |f_n| \, dm \; < \; \epsilon, \qquad \underline{\text{for all}} \quad n \geq 1.$$

Show that if $\{f_n\}_{n=1}^{\infty}$ is equi-integrable and A is a measurable set with $m(A) < \infty$ and $f_n \to f$, m-a.e. in A, then

$$\lim_{n \to \infty} \int_A f_n \, dm \; = \; \int_A f \, dm.$$

Hint: Use Egorov and Fatou.