University of Massachusetts
Department of Mathematics and Statistics
Advanced Exam in Geometry
For January, 2017

Do 5 out of the following 8 problems. Indicate clearly which questions you want graded. *Passing standard:* 70% with three problems essentially complete. **Justify all your answers.**

1. Show that the n-dimensional real projective space RP^n is orientable if and only if n is odd.

2. Let $M = N \times S^1$ be the Cartesian product of N and the circle, where N is a smooth, connected, compact manifold. Show that the Euler characteristic of M is zero, where the Euler characteristic of M, $\chi(M)$, is defined to be $\chi(M) := \sum_{k=0}^{\dim M} (-1)^k b_k(M)$. Here $b_k(M) := \dim H^k_{dR}(M)$ is the k-th Betti number.

3. Let M be a smooth, connected manifold. Show that for any two points $p, q \in M$, there is a diffeomorphism $\phi : M \to M$ which sends p to q, i.e., $\phi(p) = q$.

4. Let D be the smooth 2-dimensional distribution on $\mathbb{R}^3$ spanned by vector fields V, W, where
$$V = x\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + x(y+1)\frac{\partial}{\partial z}, \; W = \frac{\partial}{\partial x} + y\frac{\partial}{\partial z}.$$
   (a) Find a 1-form α such that $\alpha(V) = \alpha(W) = 0$.
   (b) Show that D is integrable.
   (c) Describe the integral manifold of D through the origin $(0,0,0)$.

5. Suppose that a Lie group G has a smooth left action "$\cdot$" on a smooth manifold M: $(g, x) \mapsto g \cdot x$ for all $g \in G, x \in M$.
   (a) Show that for all $x \in M$, the stabilizer
$$G_x = \{g \in G \mid g \cdot x = x\}$$
   is a Lie subgroup of G.
   (b) Let $\mathcal{X}$ be the space of smooth vector fields on M and let $\mathfrak{g}$ be the Lie algebra of G. Show that there is a unique linear transformation $X \mapsto \tilde{X}$ from $\mathfrak{g}$ to $\mathcal{X}$ such that
$$\tilde{X}_{g\cdot x} = (\varphi_x)_*(X_g)$$
   for all $g \in G$, $x \in M$, where $\varphi_x(g) = g \cdot x$.

(c) Show that the Lie algebra of G_x is isomorphic to

$$\{X \in \mathfrak{g} \mid \tilde{X}_x = 0\}.$$

6. Let (M, g) be a connected, orientable Riemannian manifold.

   (a) Define the *divergence* $\mathrm{div}_g(X)$ of a vector field X on M.

   (b) Prove that $\mathrm{div}_g(X)$ is independent of the choice of orientation.

   (c) Let $\lambda \in C^\infty(M)$ and define the metric

   $$g_\lambda(X, Y) := e^{2\lambda} g(X, Y).$$

   Show that $\mathrm{div}_{g_\lambda}(X) = \mathrm{div}_g(X) + (\dim M)(X\lambda)$.

7. Let

$$M = \{([x_0 : x_1 : x_2], t) \in RP^2 \times \mathbb{R} \mid x_0 + x_1 t + x_2 t^2 = 0\}.$$

   (a) Show that M is an embedded submanifold of $RP^2 \times \mathbb{R}$.

   (b) Let $\pi\colon M \to RP^2$ be projection onto the first factor. Find the regular values of π.

8. Let $f\colon \mathbb{R}^2 \to \mathbb{R}^3$ be given by

$$f(u, v) = (\sinh u \cos v, \sinh u \sin v, v).$$

   (a) Show that $M = f(\mathbb{R}^2) \subset \mathbb{R}^3$ is a 2-dimensional submanifold.

   (b) Compute the Gaussian curvature of M with the metric induced from $\mathbb{R}^3$.

   (c) Write the geodesic equations for M and determine if, suitably parametrized, any of the coordinate curves $\{u = \text{constant}\}$ or $\{v = \text{constant}\}$ are geodesics on M.

   (d) Draw a picture of the surface M.