DEPARTMENT OF MATHEMATICS AND STATISTICS
UNIVERSITY OF MASSACHUSETTS
ADVANCED EXAM - DIFFERENTIAL EQUATIONS

Wednesday, January 13, 2016
10:00AM – 1:00PM
LGRT 1530

Do five of the following problems. All problems carry equal weight.
Passing level: 75% with at least three substantially complete solutions.

(1) Show that the nonlinear autonomous system

$$\begin{aligned} \dot{x} &= x - y - x^3 - xy^2 \\ \dot{y} &= x + y - x^2 y - y^3 \end{aligned}$$

has a unique equilibrium point, which is unstable, and exactly one limit cycle, which is stable.
HINT: Consider the system in polar coordinates.

(2) Consider the 2×2 linear system $\dot{x} = A(t)x$ with coefficient matrix

$$A(t) = \begin{pmatrix} 0 & 1 \\ -1 - \epsilon \cos t & 0 \end{pmatrix}$$

which is equivalent to the (scalar) Mathieu's equation, $\ddot{u} + (1 + \epsilon \cos t)u = 0$; here ϵ is a prescribed constant with $0 < \epsilon < 1$.
(a) Define the monodromy matrix, M, for this 2π-periodic system, and show that $\det M = 1$.
(b) Define the Floquet multipliers, say μ_1, μ_2, associated with M, and show that they satisfy

$$\mu^2 - \tau\mu + 1 = 0, \qquad \text{where } \tau \doteq \operatorname{tr} M\,.$$

(c) Prove that *if* $|\tau| < 2$, then all solutions $x(t)$ of this system remain bounded as $t \to +\infty$. CAUTION: Do not attempt to show that $|\tau| < 2$.

(3) Develop a Lyapunov-type argument to establish that the equilibrium point $(0, 0)$ of the system

$$\begin{aligned} \dot{x} &= x^3 + x^2 y \\ \dot{y} &= -y + x^2 \end{aligned}$$

is *nonlinearly unstable*; proceed as follows.
(a) First show that linear stability analysis is inconclusive.

(b) In the wedge-shaped region, $\Omega = \{ (x, y) \in \mathbb{R}^2 \, : \, |y| < x \}$, construct a Lyapunov function, $V(x, y)$, with the properties: (i) $V(x, y) > 0$ in $\Omega \cap B_\delta(0)$, (ii) $V(x, y) = 0$ on $\partial\Omega \cap B_\delta(0)$, (iii) $\dot{V}(x, y) > 0$ in $\Omega \cap B_\delta(0)$, for some sufficiently small $\delta > 0$; here $B_\delta(0)$ denotes the disc of radius δ around the origin.
(c) Use (b) to prove that $(0, 0)$ is an unstable equilibrium point.

(4) Let u be a harmonic function in the unit ball in $\mathbb{R}^3$ minus the origin, namely, $B_1^* := \{x \in \mathbb{R}^3 \, : \, 0 < |x| < 1\}$. Show that $u(x)$ can be extended to a (regular) harmonic function in the entire ball $B_1 = \{x \in \mathbb{R}^3 \, : \, |x| < 1\}$ if and only if $\lim_{|x| \to 0} |x| u(x) = 0$.

(5) Consider the Cauchy problem for $u(x, t)$,

$$t \, u_t + x \, u_x = u, \quad u(x, 1) = v(x).$$

Describe the characteristics explicitly, solve the problem, and determine the maximal open set Ω on which the solution is uniquely defined. Also, for each finite $(x_*, t_*) \in \partial\Omega$, describe the limit

$$\lim_{y \to (x_*, t_*)} u(y), \quad \text{where} \quad y = (x, t) \in \Omega,$$

if it exists. Assume $v(x) \to v_\pm$ as $x \to \pm\infty$, respectively.

(6) Consider Schrödinger's equation for a (complex valued) wave function,

$$i \, \frac{\partial \psi}{\partial t} = -\Delta\psi + V(x)\, \psi, \quad \text{in} \quad \Omega \times (0, \infty),$$

with $\psi = 0$ on $\partial\Omega \times (0, \infty)$, and where the potential $V(x)$ is real valued. Show that $\int_\Omega |\psi(x, t)|^2 \, dx$ is constant, and use this to prove uniqueness of solutions to the initial value problem $\psi(x, 0) = \psi^0(x)$.

(7) Suppose that $u(x, t)$ solves the heat equation

$$u_t - \Delta u = 0 \quad \text{in} \quad \mathbb{R}^n \times (0, \infty),$$

and suppose that the long-time limit of $u(x, t)$ is $w(x)$, that is, for any compact $K \subset \mathbb{R}^n$,

$$\sup_{x \in K} |u(x, t) - w(x)| \to 0 \quad \text{as} \quad t \to \infty.$$

Show that w is harmonic.
HINT: for any test function $\phi \in C_c^\infty(\mathbb{R}^n)$, consider the integral

$$I[t_1, t_2] = \int_{t_1}^{t_2} \int_{\mathbb{R}^n} \big(u(x, t) - w(x)\big) \, \Delta\phi(x) \, dx \, dt.$$