DEPARTMENT OF MATHEMATICS AND STATISTICS
UNIVERSITY OF MASSACHUSETTS, AMHERST

ADVANCED EXAM — ALGEBRA

JANUARY 2016

Passing Standard: To pass the exam it is sufficient to solve five problems including a least one problem from each of the three parts. Show all your work and justify your answers carefully.

1. Group theory and representation theory

1. Determine the character table of the group

$$Q = \{\pm 1, \pm i, \pm j, \pm k\}$$

with $i^2 = j^2 = k^2 = -1$ and $ij = k$.

2. Let $p < q$ be primes and G a group of order pq^n. Show that G is solvable, that is, there exists subgroups N_i such that $G = N_0 \trianglerighteq N_1 \trianglerighteq \cdots \trianglerighteq N_r = (e)$ such that N_{i-1}/N_i is abelian.

3. Let G be a free abelian group of rank $r \geq 1$, so G is isomorphic to $\mathbf{Z}^r$ as groups. Fix $n \geq 1$. Show that G has only finitely many subgroups of index n.

2. Commutative algebra

4. Let R be a commutative ring and fix a nonzero element $\alpha \in R$.

(1) Give a precise definition of the localization $R_{(\alpha)}$ obtained by inverting α.
(2) Give an example of a ring R and a nonzero $\alpha \in R$ such that the natural map $R \to R_{(\alpha)}$ is not injective.
(3) Prove that there is a bijective correspondence between the ideals of $R_{(\alpha)}$ and the ideals of R not containing α.

5. Let p be a prime and set $R = \mathbf{Z}/p^n$ for some $n \geq 1$. Let A, B, C be finitely generated R-modules such that $A \oplus C \cong B \oplus C$. Prove that $A \cong B$.

6. Let R be a reduced (that is, R has no non-zero nilpotent elements) commutative nonzero ring that has a unique prime ideal. Show that R is a field.

3. Field theory and Galois theory

7. Give an example, with proof, of a Galois extension L/K with $\mathrm{Gal}(L/K) \cong D_5$, the dihedral group of order 10 of symmetries of a regular pentagon.

8. Let K be a field and suppose that $f \in K[x]$ is irreducible of degree n. Let L/K be an extension such that $[L : K] = m$ is relatively prime to n. Prove that f is irreducible in $L[x]$.

9. Let L/K be an extension of fields and let R be a ring such that $K \subseteq R \subseteq L$.

(1) If L/K is a finite extension, prove that R is a field.
(2) Give an example where L/K is infinite and R is not a field.