DEPARTMENT OF MATHEMATICS AND STATISTICS
UNIVERSITY OF MASSACHUSETTS AMHERST
BASIC NUMERICAL ANALYSIS EXAM
JANUARY 2014

Do five of the following problems. All problems carry equal weight.
Passing level:

Masters: 60% with at least two substantially correcct

PhD: 75% with at least three substantially correct.

1. Find the quadrature point x_0 so that the scheme

$$\int_0^6 f(x)\,dx \approx \omega_1 f(0) + \omega_2 f(1) + \omega_3 f(x_0)$$

has the highest possible degree of precision. You do not need to give $\omega_1, \omega_2, \omega_3$.

2. Consider the family of linear multistep methods, for solving $y' = f(y)$, at evenly spaced points,

$$y_{n+1} = \alpha y_n + \frac{h}{2}(2(1-\alpha)f(y_{n+1}) + 3\alpha f(y_n) - \alpha f(y_{n-1})),$$

where $h = x_n - x_{n-1}$ is the grid spacing. Analyze consistency and order of the methods as functions of α, determining the value α^* for which the resulting method has maximal order.

3. Consider solving

$$\begin{bmatrix} 8 & 2 & 0 \\ 1 & 4 & 2 \\ 0 & 3 & 5 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},$$

using Jacobi iteration. Starting from an initial guess $\mathbf{x}^0 = 0$, find $\mathbf{x}^1$ and $\mathbf{x}^2$. Find the rate of convergence.

4. Let

$$H = \begin{bmatrix} 1 & 1/2 & 1/3 \\ 1/2 & 1/3 & 1/4 \\ 1/3 & 1/4 & 1/5 \end{bmatrix}.$$

(a) Find the decomposition $H = LDL^T$, where D is diagonal and L is lower triangular with 1's on the diagonal.

(b) Give an example of a 3×3 symmetric matrix M which **cannot** be decomposed this way.

5. One wants to solve the equation $x + \ln(x) = 0$, whose root is $r \approx .5$, using one of the following fixed point methods,

(a)

$$x_{k+1} = -\ln(x_k)$$

(b)

$$x_{k+1} = e^{-x_k}$$

(c)

$$x_{k+1} = \frac{x_k + e^{-x_k}}{2}$$

Which of the above methods can be used? Which of the above methods is the best method assuming that the initial guess x_0 is close enough to the root?

6. Let

$$f(x) = \cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + ...$$

Consider the following rational approximation

$$r(x) = \frac{a_0 + a_2 x^2 + a_4 x^4}{1 + b_2 x^2}.$$

Determine the coefficients of r in such a way that

$$f(x) - r(x) = \gamma_8 x^8 + \gamma_{10} x^{10} + ...$$

7. Find the value of α that minimizes

$$\int_0^1 (x^3 - \alpha x^2)^2 dx$$