DEPARTMENT OF MATHEMATICS AND STATISTICS
UNIVERSITY OF MASSACHUSETTS
BASIC EXAM - STATISTICS
WEDNESDAY, JANUARY 15, 2014

Do all five problems. Sixty points are needed to pass at the Master's level and seventy-five at the Ph.D. level.

1. (10 points) Let $\overline{X}_1$ and $\overline{X}_2$ be sample means based on two independent samples of sizes n_1 and n_2, taken from two populations with common unknown mean μ, and with known variances σ_1^2 and σ_2^2, respectively.

   (a) Find the minimum-variance unbiased estimator of μ in the family of estimators that are linear combinations of $\overline{X}_1$ and $\overline{X}_2$. Be sure to show how you derive this estimator.

   (b) Find the variance of the estimator in (a).

2. (10 points) Let $\widehat{p}_n$ be the MLE of the MLE of the p based on n i.i.d. Bernoulli variables with probability p of success. We are particularly interested in the logistic transformation

$$\theta = \log_e\left(\frac{p}{1-p}\right), \text{ and } \widehat{\theta}_n = \log_e\left(\frac{\widehat{p}_n}{1-\widehat{p}_n}\right)$$

   (a) In terms of n and p, state with brief explanation the approximate distribution of $\widehat{\theta}_n$.

   (b) Obtain an approximate 95% confidence interval for θ for a fixed large n.

3. (50 points) Let $X_1, \ldots X_n$ be i.i.d. each distributed Poisson(λ); that is $P(X_i = x) = \lambda^x e^{-\lambda}/x!$ for $x = 0, 1, 2, \ldots$, and $\lambda > 0$. Recall that $E(X_i) = \lambda$, $V(X_i) = \lambda$.

   (a) Write down the likelihood function for λ and find the maximum likelihood estimator. Justify that your solution maximizes the likelihood function

   (b) Find $I_1(\lambda)$, the information for λ in each X_i and then the total information $I(\lambda)$.

   (c) Find the Cramer-Rao lower bound for unbiased estimators of λ. Using this result, will you conclude or deny that the MLE is a UMVUE for λ? Explain why.

   (d) Give the asymptotic distribution of the MLE $\hat{\lambda}$ (properly centered and scaled).

   (e) Use the previous part and whatever other justification is needed to develop an approximate confidence interval for λ.

(f) Let $\theta = \Pr\{X > 0\}$, where X has the Poisson distribution as stated at the beginning of the problem.

i. Find the MLE of θ, and show that (with explanations) it is consitent.

ii. Is the MLE you found in previous question unbiased? Explain your answer.

(g) Consider making inferences for λ within the Bayesian framework. Suppose λ has a prior distribution which is gamma with parameters α and β [i.e., $\pi(\lambda) = \lambda^{\alpha-1}e^{-\lambda/\beta}/(\beta^\alpha\Gamma(\alpha))$]. Note that $E(\lambda) = \alpha\beta$ and $\mathrm{Var}(\lambda) = \alpha\beta^2$.

i. Find the Posterior distribution of λ (It should be represented in terms of a known distribution).

ii. Find the posterior mean and posterior variance of λ

iii. Describe how to construct a 95% equal-tail posterior interval for λ.

4. (15 points)

(a) State carefully the Neyman-Pearson lemma for testing a simple null hypothesis against a simple alternative hypothesis.

(b) Based on one observation X from a distribution with p.d.f. $f(x)$, derive the most powerful size 0.05 test for testing

$$H_0: \ f(x) = 2x, \quad \text{for } 0 < x < 1$$

against

$$H_1: \ f(x) = 1, \quad \text{for } 0 < x < 1.$$

Be sure to give the critical region explicitly.

(c) Compute the power of the test.

5. (15 points) Let $X_1, \cdots, X_n$ be a random sample from a normal distribution with mean μ and standard deviation σ, where both μ and σ are unknown. Consider testing

$$H_0: \ \mu = \mu_0 \ \text{ against } \ H_1: \ \mu \neq \mu_0,$$

with μ_0 being a given number.

(a) Derive (step by step) the size α likelihood ratio test with specification of the critical value and critical region for the test in term of a well-known distribution.

(b) Explain how to construct 95% confidence interval for μ using the distribution relating to part (a).