DEPARTMENT OF MATHEMATICS AND STATISTICS
UNIVERSITY OF MASSACHUSETTS AMHERST
BASIC NUMERICAL ANALYSIS EXAM
JANUARY 2014

Do five of the following problems. All problems carry equal weight.
Passing level:
Masters: 60% with at least two substantially correcct
PhD: 75% with at least three substantially correct.

1. Consider the equation
$$\dot{y} = f(x, y).$$
Find a_1, a_2, b_1, and b_2 so that the discretization
$$y_{n+1} = a_1 y_{n-1} + a_2 y_{n-2} + h(b_1 f(x_{n-1}, y_{n-1}) + b_2 f(x_{n-2}, y_{n-2}))$$
is of maximal order.

2. Let A be an n by n matrix with real, positive spectrum. Prove the following, or give a counterexample.
   (a) A is positive definite.
   (b) The Gauss-Seidel iteration applied to $Au = f$ always converges.

3. Consider the integral $I(f) = \int_{-2}^{2} f(x)\,dx$. Find a, b, c so that the quadrature rule
$$I_h(f) = af(-1) + bf(0) + cf(1)$$
has maximal order. What is the degree of precision?

4. Consider approximating the second derivative $f''(x)$ using a numerical differentiation formula $D_h^{(2)} f = af(x) + bf'(x) + cf(x+h) + df'(x+h)$. Find a, b, c, d and compute the truncation error. Assume that the values of f and f' are evaluated up to a precision of ϵ, what is the effect of this roundoff error on the error in approximating $f''(x)$?

5. Consider the fixed point iteration $x_{n+1} = g(x_n)$ for the function $g(x) = \cos(e^{-x^2})$. Show that there is a unique root and that for any starting value $x_0 \in \mathbb{R}$, the iteration converges. (Hint: $2|x| \leq e^{x^2}$ for all $x \in \mathbb{R}$.)

6. Let A be a nonsingular matrix of order n, I be the identity matrix of order n and X_k, $k = 0, 1, ...,$ be a sequence of matrices(order n) satisfying
$$X_{k+1} = X_k + X_k(I - AX_k).$$
   (a) Show that the inverse of A, A^{-1}, is a fixed point of the above iteration.
   (b) Assume that $\|I - AX_0\| < 1$, show that $E_k = I - AX_k$ converges quadratically to zero(Hint: Multiply the above equation by A and subtract both sides from I.)
   (c) What is X_k converging to?

7. Suppose the function $f(x) = \cos(x)$ is interpolated at x_i, $i = 0, 1, ..., n$, on the interval $[0, 6\pi]$ by a polynomial $p_n(x)$ whose degree does not exceed n. Show that $|f(x) - \cos(x)|$ approaches 0 as n approaches ∞, where x lies in $[0, 6\pi]$.