DEPARTMENT OF MATHEMATICS AND STATISTICS
UNIVERSITY OF MASSACHUSETTS
MASTER'S OPTION EXAM-APPLIED MATHEMATICS
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Do five of the following problems. All problems carry equal weight. Passing level: 60% with at least two substantially correct.

1. For the transcritical bifurcation:
   - Write the normal form.
   - Sketch the bifurcation diagram (using solid lines for stable fixed points and dashed lines for unstable ones).
   - For a field
   $$\dot{x} = f(x, r)$$
   provide the conditions near a critical point $(x^\star, r_c)$ under which we can consider a bifurcation to be transcritical.
   - Apply these conditions to the problem (and determine its bifurcation diagram)
   $$\dot{x} = x(r - e^x)$$

2. Consider the weakly perturbed oscillator:
   $$\ddot{x} + x = -2\epsilon\dot{x}$$
   - Solve the system exactly with initial conditions $x(0) = 0$ and $\dot{x}(0) = 1$.
   - Attempt to solve it by regular perturbation theory $x(t) = x_0(t) + \epsilon x_1(t) + \ldots$ and obtain the first two orders ($x_0(t)$ and $x_1(t)$). Explain the problem arising. Hint: For the inhomogeneous solution in $x_1(t)$, try $x_1 = Ct\sin(t)$.
   - Try to solve for the leading order x_0, by the same perturbative expansion, using *also* two-timing $\tau = t$ and $T = \epsilon t$. Does that give a better approximation and why ?

3. (a) Consider the problem $u_{tt} = c^2 u_{xx}$ in the *infinite line* with the initial conditions: $u(x,0) = \phi(x) = 0$, $u_t(x,0) = \psi(x) = xe^{-x^2}$.
   (a) Find the solution to this PDE at all times.
   (b) Sketch the solution of the PDE at $t = 0$ and at a large time $t >> 0$.

   (b) Solve the heat equation with convection:

$$u_t - ku_{xx} + Vu_x = 0$$

   for $-\infty < x < \infty$ and $u(x,0) = \phi(x)$.

4. Solve the 2d Laplace equation

$$u_{xx} + u_{yy} = 0$$

in the disk $r < a$ with the boundary condition

$$u = 1 + 3\sin(\theta)$$

on the boundary of the domain at $r = a$. Show all the details of your calculation.

5. Solve the equation

$$u_t = ku_{xx}$$

with boundary condition $u(0,t) = u(L,t) = 0$ and initial condition $u(x,0) = x$. Then use Parseval's identity for the Fourier series of $f(x) = x$ to compute $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

6. (a) Show that the eigenvalue problem

$$\begin{cases} -f''(x) & = \lambda f(x) \\ f'(0) & = a_0 f(0), \quad f'(L) = -a_L f(L), \end{cases}$$

   cannot have negative eigenvalues when $a_0 > 0$ and $a_L > 0$.

   (b) Again, show that λ cannot be negative for

$$\begin{cases} f''''(x) = \lambda f(x) \\ f(0) = f(L) = f''(0) = f''(L) = 0 \end{cases} .$$

7. Consider the rabbit-sheep problem for $x > 0$ and $y > 0$:

$$\dot{x} = x(3 - x - y)$$

$$\dot{y} = y(2 - x - y)$$

(a) Find the fixed points.

(b) Classify their stability and sketch the phase plane.

(c) Explain why there can not be any limit cycles in this system.