Department of Mathematics and Statistics
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ADVANCED EXAM — DIFFERENTIAL EQUATIONS
JANUARY 2014

Do five of the following seven problems. All problems carry equal weight. Passing level: 75% with at least three substantially complete solutions.

1. Consider the two-dimensional dynamical system:

$$\begin{aligned} \dot{x} &= y\,, \\ \dot{y} &= -\sinh x\,. \end{aligned}$$

   (a) Discuss the linear stability of the equilibrium point $(x, y) = (0, 0)$.

   (b) Prove the nonlinear stability of $(0, 0)$ by constructing a Lyapunov function.

   (c) Is the origin asymptotically stable? Why or why not?

2. Suppose that two solutions, $u_1(x,t)$ and $u_2(x,t)$, of the wave equation in R^1,

$$\frac{\partial^2 u}{\partial t^2} - \frac{1}{c^2}\frac{\partial^2 u}{\partial x^2} = f(x,t)\,,$$

satisfy

$$u_1(x,0) = u_2(x,0)\,, \qquad \sup_x \left| \frac{\partial u_1}{\partial t}(x,0) - \frac{\partial u_2}{\partial t}(x,0) \right| \le \epsilon\,,$$

for $\epsilon > 0$.

(a) Prove that

$$\sup_x \left| \frac{\partial u_1}{\partial t}(x,t) - \frac{\partial u_2}{\partial t}(x,t) \right| \le \epsilon\,, \qquad \text{for all} \quad t > 0\,.$$

(b) Provide an estimate in terms of ϵ and t for

$$\sup_x |\, u_1(x,t) - u_2(x,t)\,|\,, \qquad \text{for all} \quad t > 0\,.$$

3. The following third-order ODE arises in the boundary-layer theory of fluid mechanics:

$$\frac{d^3u}{dx^3} + u\,\frac{d^2u}{dx^2} = 0 \qquad \text{in} \quad 0 < x < +\infty.$$

Consider a solution $u(x)$ satisfying the conditions

$$u(0) = 0\,, \;\; \frac{du}{dx}(0) = 0\,, \;\; \frac{d^2u}{dx^2}(0) = 1\,.$$

(a) Show that the second derivative, $\phi(x) = \partial^2 u/dx^2$, of the solution $u(x)$ satisfies the integral equation

$$\phi(x) \;=\; \exp\left(-\frac{1}{2}\int_0^x (x-y)^2 \phi(y)\, dy \right)\,.$$

HINT: express $u(x)$ as an integral involving ϕ.
(b) Using (a), give a proof that the solution $u(x)$ exists on the entire interval $0 < x < +\infty$.

4. Consider the initial-boundary-value problem

$$\begin{aligned} \frac{\partial u}{\partial t} &= \triangle u + \alpha u && \text{in } \Omega \times [0,\infty)\,, \\ u &= 0 && \text{on } \partial\Omega \times [0,\infty)\,, \\ u &= u_0 \in L^2(\Omega) && \text{at } t = 0\,. \end{aligned}$$

Ω is a smooth bounded domain in R^n, and α is a positive constant.
(a) Suppose that $\alpha < \lambda_1(\Omega)$, the smallest eigenvalue of $-\triangle$ on Ω. Show that the $L^2(\Omega)$-norm of $u(x,t)$ decreases to zero: namely,

$$\int_\Omega u(x,t)^2\, dx \;\to 0 \qquad \text{as } \; t \to +\infty.$$

(b) Suppose now that instead $\lambda_1(\Omega) < \alpha < \lambda_2(\Omega)$, where $\lambda_2(\Omega)$ is the second eigenvalue. What is the behavior of the $L^2(\Omega)$-norm of $u(x,t)$ in that case?

5. Consider the system

$$\dot{x} = -x + y\, g(\sqrt{x^2+y^2}), \qquad \dot{y} = -y - x\, g(\sqrt{x^2+y^2}),$$

where $g(r) = 1/\log r$ for $r > 0$ and $g(0) = 0$.

(a) Linearize around the equilibrium and show that the origin of the linearization is a stable node.

(b) Show directly that the origin is a stable focus for the nonlinear system.

(c) Why do (a) and (b) not yield a contradiction?

6. Consider harmonic functions, $u(x, y)$, in the infinite strip:

$$\Omega \;=\; \{\,(x,y)\;:\; -\infty < x < +\infty\,,\;\; -\frac{\pi}{2} < y < +\frac{\pi}{2}\;\}\,.$$

(a) Construct explicitly a function $w(x, y)$ satisfying $\triangle w = 0$ in Ω and $w = 0$ on $\partial\Omega$, for which $w > 0$ in Ω.
(b) Suppose that $u(x, y)$ is *any* smooth solution of $\triangle u = 0$ in Ω with $u = 0$ on $\partial\Omega$. Show that if

$$\lim_{|x|\to\infty}\; \sup_{|y|\le \pi/2}\; e^{-|x|}\,|u(x,y)| \;=\; 0\,,$$

then u must be identically zero in Ω.
HINT: Use a maximum principle argument on large rectangles that approach Ω, and compare u to small multiples of w.

7. Consider the linear system

$$\dot{X} = A(t)\,X, \quad \text{where} \quad A(t) = \begin{pmatrix} 1 + \frac{\cos t}{2+\sin t} & 0 \\ 1 & -1 \end{pmatrix}.$$

(a) Write down a fundamental matrix solution $\Phi(t)$ for the system. HINT: The first equation decouples.

(b) Find a constant matrix B and periodic matrix $P(t)$ such that

$$\Phi(t) = P(t)\, e^{t\,B}.$$