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Advanced Exam – Algebra. January 13, 2014.

Passing Standard: It is sufficient to do five problems correctly, including at least one problem from each of the three parts.

1. Group Theory and Representation Theory

1. Let G be a group given by generators and relations, as follows:

$$G = \langle a, b \mid a^4 = b^5 = 1,\ aba^{-1} = b^2 \rangle.$$

(a) Show that G is a finite group.
(b) Find dimensions of all irreducible representations of G.

2. Show that $(\mathbb{R}, +)$ (the group of real numbers with operation addition) is isomorphic to a direct sum of infinitely many infinite groups.

3. Let G be finite group with the property that for every non-trivial normal subgroup K of G, G/K is abelian. Let $\rho : G \to \mathrm{GL}_n(\mathbb{C})$ be an irreducible representation of G with $n > 1$. Show that ρ is injective.

2. Commutative Algebra

4. Let $I \subset \mathbb{Z}[i]$ (resp. $J \subset \mathbb{Z}[i]$) be a principal ideal generated by $7 - i$ (resp. by $-7 + 6i$). Let A be the $\mathbb{Z}[i]$-module $(\mathbb{Z}[i]/I^{2014}) \otimes_{\mathbb{Z}[i]} (\mathbb{Z}[i]/J^{2014})$.

(a) Find the number of elements in A.
(b) Describe A as an abelian group (decompose it into a direct sum of cyclic groups).

5. Let R be a non-zero commutative ring and let M be a non-zero simple R-module. (Recall that this means that M has no R-submodules other than 0 and M.)

(a) Prove that M is isomorphic to $R/\mathfrak{m}$ for some maximal ideal $\mathfrak{m}$ of R.
(b) Prove that if $\varphi : M \to M'$ is a R-module homomorphism to another simple R-module M', then φ is either the zero map or an isomorphism.

6. Consider the $\mathbb{C}$-algebra homomorphism

$$f : \mathbb{C}[x, y] \to \mathbb{C}[t]$$

given by $f(x) = t^2$ and $f(y) = t^2 - t$. Show that the kernel of f is a principal ideal.

3. Field Theory and Galois Theory

7. Is the polynomial $x^4 + 4x^3 + 6x^2 + 2x + 1$ irreducible over
(a) $\mathbb{Q}[x]$?
(b) $\mathbb{F}_3[x]$?

8. Let $f(x) \in \mathbb{Q}[x]$ be quadratic and monic, say $f = x^2 + bx + c$. Let K denote the splitting field of the quartic polynomial $f(x^2)$. Show that the Galois group $\mathrm{Gal}(K/\mathbb{Q})$ is a subgroup of the dihedral group with 8 elements (the group of symmetries of a square).

9. Fix a field K and an irreducible separable polynomial $f(x) \in K[x]$. Let L denote the splitting field of f over K and let $\alpha_1, \ldots, \alpha_n \in L$ denote the roots of f. Assume that $\mathrm{Gal}(L/K) \cong S_n$ the group of permutations on n elements.
(a) Show that the fields $K(\alpha_1), \ldots, K(\alpha_n)$ are distinct.
(b) Let $\sigma \in \mathrm{Gal}(L/K)$ send the field $K(\alpha_1)$ to itself. Show that σ fixes all elements of the field $K(\alpha_1)$.