DEPARTMENT OF MATHEMATICS AND STATISTICS
UNIVERSITY OF MASSACHUSETTS
BASIC EXAM - STATISTICS
Friday, January 18, 2013

Work all five problems. Sixty points are needed to pass at the Master's level and seventy-five at the Ph.D. level.

1. (**20** points) Let $X_1, \ldots X_n$ be a random sample from the exponential distribution, with unknown location parameter θ and scale parameter λ, whose density is given by

$$p(x|\theta, \lambda) = \lambda \exp[-\lambda(x-\theta)], \quad \theta \le x < \infty,$$

where $-\infty < \theta < \infty$, and $0 < \lambda < \infty$.

   (a) Find the mean and variance of $(X - \theta)$.
   (b) Find the mean (μ) and variance (σ^2) of X.
   (c) Find the MLEs of θ and λ. (NOTE: You must justify why your answers are MLEs.)
   (d) Write down the MLEs of μ and σ^2, with brief explanations.

2. (**20** points) Let $X_1, \ldots X_n$ denote a random sample from the $Poisson(\lambda)$ distribution. Let $\theta = \lambda^2$.

   (a) Determine the Cramer-Rao bound on an unbiased est imator of λ.
   (b) Determine the Cramer-Rao bound on an unbiased est imator of $\theta = \lambda^2$.
   (c) Determine the Uniform Minimum Variance Unbiased Estimator of $\theta = \lambda^2$.
   (d) Consider

$$\tilde{\theta}_n = \frac{1}{n} \sum_{k=1}^{n} X_i(X_i - 1)$$

   as an estimator of $\theta = \lambda^2$. Discuss its properties, good features, shortcomings, etc., being sure to include at least one good feature and at least one shortcoming.

3. (**20** points) Let $X_1, \ldots X_n$ be a random sample from a gamma distribution with probability density function:

$$f(x|\alpha, \beta) = \frac{x^{\alpha-1} \exp(-x/\beta)}{\Gamma(\alpha)\beta^\alpha}, \quad x > 0, \ \alpha > 0, \ \beta > 0.$$

Consider the prior distribution for the parameter β as an inverse gamma distribution with the following probability density function [i.e., $\beta \sim$Inverse Gamma(λ, θ)]:

$$p(\beta|\lambda, \theta) = \frac{\exp(-1/(\theta\beta))}{\Gamma(\lambda)\theta^\lambda \beta^{\lambda+1}}, \quad \beta > 0, \ \lambda > 0, \ \theta > 0.$$

Note that for $\beta \sim$Inverse Gamma(λ, θ), the following are known

$$E(\beta) = \left[\frac{1}{\theta(\lambda - 1)}\right], \text{ for } \lambda > 1,$$

$$\text{Var}(\beta) = \left[\frac{1}{\theta^2(\lambda-1)^2(\lambda-2)}\right], \text{ for } \lambda > 2,$$

(a) Find the posterior distribution of β.

(b) Find the posterior mean of β.

(c) Describe how to construct a 95% equal-tail posterior interval for β.

4. (**20** points) Let $X_1, \dots X_n$ be a random sample from a normal distribution with mean θ and variance σ^2. Suppose we want to test the null hypothesis $H_0 : \theta \leq 0$ against the one-sided alternative $H_1 : \theta > 0$.

   For the Bayesian parts of the problem, the prior distribution on θ is normal with mean 0 and variance τ^2, with τ^2 known. Note that this prior is symmetric about the hypotheses such that $P(\theta \leq 0) = P(\theta > 0) = 0.5$.

   (a) Find the posterior distribution of θ.

   (b) Find the posterior probability that H_0 is true, $P(\theta \leq 0|x_1, x_2, \dots, x_n)$.

   (c) Find an expression for the p-value corresponding to a value of $\bar{x}$, using tests that reject for large values of $\bar{X}$.

   (d) For the special case $\sigma^2 = \tau^2 = 1$, compare $P(\theta \leq 0|x_1, x_2, \dots x_n)$ and the p-value for values of $\bar{x} > 0$. Show that the Bayes probability is always greater than the p-value.

   (e) Using the expression derived above, show that

$$\lim_{\tau^2 \to \infty} P(\theta \leq 0|x_1, \dots x_n) = \text{ p-value.}$$

5. (**20** points) Let $X_1, \dots X_n$ be a random sample from a $N(\mu, 1)$ distribution, where the mean μ is unknown, which has pdf:

$$f(x|\mu) = \frac{1}{\sqrt{2\pi}} \exp\{-\frac{1}{2}(x - \mu)^2\}, \text{ for } -\infty < x < \infty, -\infty < \mu < \infty.$$

   (a) Consider testing

$$H_0 : \ \mu = 0 \ \text{ versus } H_1 : \ \mu = \mu_1,$$

   where $\mu_1 > 0$ is a given number.

      i. Derive the most powerful test (MPT). You need to specify the critical region for the test of size α, with critical value being a quantile of a standard normal distribution (with justifications).

      ii. Find the power of the MPT when $\mu = 1/2$ for $n = 16$ and $\alpha = 0.05$. (A table of the $N(0, 1)$ distribution is enclosed.)

   (b) Now consider another problem of testing

$$H_0 : \ \mu = 0 \ \text{ versus } H_1 : \ \mu \neq 0.$$

      i. Derive the likelihood ratio test (LRT). You need to specify the critical region for the test of size α, with critical value also being a quantile of the standard normal distribution (with justifications).

      ii. Find the power of the LRT when $\mu = 1/2$ for $n = 16$ and $\alpha = 0.05$.

   (c) Based on the power computations, can the above LRT be a UMP test? Explain why.