BASIC EXAM – ADVANCED CALCULUS & LINEAR ALGEBRA
DEPARTMENT OF MATHEMATICS & STATISTICS
UNIVERSITY OF MASSACHUSETTS, AMHERST

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Provide solutions for **seven** of the following ten problems. Each problem is worth 10 points. To pass at the Master's level, it is sufficient to have 42 points (60%), with 3 essentially correct solutions (including at least one from each part); 53 points (75%) with at least two essentially complete solutions from each part is sufficient for passing at the Ph.D. level. Indicate clearly which problems you want graded. Be sure to show all your work.

Part I. Linear Algebra

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(1) Determine the Jordan canonical form of the matrix $\begin{bmatrix} -7 & 9 \\ -4 & 5 \end{bmatrix}$.

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(2) Let $T : V \to V$ be a linear transformation of a four-dimensional real vector space V. Assume that the characteristic polynomial of T is $x^4 - 3x^3$.
   (a) Show that V has T-invariant subspaces of dimension 1, 2 and 3.
   (b) What can you say about the rank of T?

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(3) Let M denote the four-dimensional vector space of 2×2 complex matrices. Let $A \in \mathrm{GL}_2(\mathbb{C})$ be a diagonalizable matrix with eigenvalues α and β. Let $T : M \to M$ be the linear transformation given by $T(X) = AXA^{-1}$. Determine the eigenvalues of T. (Hint: start by considering diagonal X.)

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(4) Recall that the dual V^* of a finite-dimensional vector space V over a field F is the space of all linear maps $V \to F$; it has the same dimension over F as V, a fact you may assume without proof.

If V, W are finite-dimensional vector spaces over F and $f : V \to W$ is an F-linear map, then we define $f^* : W^* \to V^*$ by $f^*(g) = g \circ f$ for all $g \in W^*$.
   (a) Prove that f^* is a linear map.
   (b) Prove that f is an isomorphism if and only if f^* is an isomorphism.

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(5) Find an orthogonal basis for the bilinear form over $\mathbb{R}$ given by $(\mathbf{x}, \mathbf{y}) \mapsto \mathbf{x}^{\mathrm{t}} A \mathbf{y}$ where

$$A = \begin{bmatrix} 1 & 4 & 4 \\ 4 & 4 & 10 \\ 4 & 10 & 16 \end{bmatrix}.$$

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(6) State and prove the Alternating Series Test for series of real numbers.

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(7) Suppose I is an interval in $\mathbb{R}$ and $f : I \to \mathbb{R}$ is a uniformly continuous function on I. Suppose further that for all $x \in I$, $|f(x)| \geq \epsilon$ where $\epsilon > 0$ is a fixed positive constant. Show that $g(x) = 1/f(x)$ is also uniformly continuous on I.

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(8) Suppose $f : \mathbb{R} \to \mathbb{R}$ is a differentiable function satisfying: $f(0) = 1$ and $|f'(x)| \leq 1$ for all $x \in \mathbb{R}$. Prove that $|f(x)| \leq |x| + 1$ for all $x \in \mathbb{R}$.

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(9) For an orientable surface S with boundary ∂S and a fixed vector $\mathbf{v} = (a, b, c)$, prove that

$$2 \iint_S \mathbf{v} \cdot \mathbf{n} \, dS = \int_{\partial S} (\mathbf{v} \times \mathbf{r}) \cdot dS$$

where $\mathbf{r} = (x, y, z)$ and $\mathbf{n}$ is the unit normal vector for S.

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(10) Compute

$$\int_C x^3 dy - y^3 dx$$

where C is a counterclockwise path traversing the unit circle once.
