UNIVERSITY OF MASSACHUSETTS
Department of Mathematics and Statistics
Basic Exam - Statistics
Tuesday, January 17, 2012

Work all problems. 60 points are needed to pass at the Masters Level and 75 to pass at the Ph.D. level.

1. Let $X_1, X_2, \ldots, X_n$ be a random sample from a distribution with the following density:

$$f(x \mid \theta) = \binom{n}{x} \theta^x (1-\theta)^{n-x}, \ for \ x = 0, 1, \ldots, n;$$

where $0 \leq \theta \leq 1$ and n is any positive integer.

   (a) Derive the MLE, $\hat{\theta}$, of θ. It is required to justify that your answer is indeed an MLE.
   (b) Is $\hat{\theta}$ an unbiased estimator? Justify your answer.
   (c) What is the variance of the estimator that you found in part a)?
   (d) Assuming $0 < \theta < 1$, what is the approximate distribution of the estimator that you found in a) as n gets large?
   (e) Use your result from part d) to find an approximate 95% confidence interval for θ.
   (f) Develop the likehood ratio test for testing

$$H_0 : \ \theta = \theta_0 \ \text{ against } \ H_1 : \ \theta = \theta_1$$

   where $0 < \theta_0 < \theta_1 < 1$. You must reduce the test in term of a statistic with known distribution, then describe the size α rejection region using this known distribution.

2. Let $Y_1, \ldots, Y_n$ be a random sample from Bernoulli(θ), and define $X = \sum_{i=1}^{n} Y_i$. Thus, $X \mid \theta \sim$ Binomial(θ), and the sampling density for X is

$$f(x \mid \theta) = \binom{n}{x} \theta^x (1-\theta)^{n-x}, \ for \ x = 0, 1, \ldots, n$$

where $0 \leq \theta \leq 1$ and n is any positive integer.

Let the prior distribution for the parameter θ be the Beta distribution with the following probability density function. (i.e. $\theta \sim \ Beta(\alpha, \beta)$) :

$$f(\theta \mid \alpha, \beta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \theta^{\alpha-1} (1-\theta)^{\beta-1}$$

where $0 \leq \theta \leq 1$, $\alpha > 0$ and $\beta > 0$.

Note that for $\theta \sim Beta(\alpha, \beta)$, $E(\theta) = \frac{\alpha}{\alpha+\beta}$ and $Var(\theta) = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$.

(a) Find the posterior distribution of θ.

(b) Find the posterior mean of θ.

(c) Describe how to construct a 90% equal-tail posterior interval for θ.

3. Let $Y_1, \ldots, Y_n$ be independent random variables with the following probability mass function

$$P(Y_i = y \mid \beta) = \frac{(\beta x_i)^y e^{-\beta x_i}}{y!}, \quad y = 0, 1, ...$$

where $x_1 < x_2 < \ldots < x_n$ are fixed, positive constant, and β is an unknown positive constant.

(a) Find the maximum likelihood estimator (MLE) of β, $\hat{\beta}_{MLE}$.

(b) Find the least squares estimator (LSE) for β, $\hat{\beta}_{LSE}$, that minimizes the sum of squares of difference between Y_i and βx_i.

(c) Find the means and variances of $\hat{\beta}_{MLE}$ and $\hat{\beta}_{LSE}$.

(d) Find the Rao-Cramer lower bound for an unbiased estimator of β.

(e) Which estimator between $\hat{\beta}_{MLE}$ and $\hat{\beta}_{LSE}$ is preferred? Justify your choice.

4. Let $X_1, \ldots, X_n$ be a random sample from the gamma distribution

$$f(x; \theta) = \theta e^{-\theta x}$$

where $x \geq 0$ and $\theta > 0$.

(a) Find the maximum likelihood estimator (MLE) for θ, $\hat{\theta}_{MLE}$.

(b) Find the asymptotic distribution of $\hat{\theta}_{MLE}$.

(c) Describe how one can construct a 95% confidence interval for θ using the likelihood ratio statistic.

Consider the prior distribution for the parameter θ as an exponential distribution,

$$\pi(\theta; \tau) = \tau e^{-\tau\theta},$$

where $\theta > 0$ and $\tau > 0$.

(e) Find the Bayes estimator of θ and show that it is a weighted average of the prior mean for θ and $\hat{\theta}_{MLE}$

(f) Describe how one can construct a 95% Bayes credible interval for θ.