DEPARTMENT OF MATHEMATICS AND STATISTICS
UNIVERSITY OF MASSACHUSETTS
MASTER'S OPTION EXAM-APPLIED MATHEMATICS
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Do five of the following problems. All problems carry equal weight.
Passing level: 60% with at least two substantially correct.

1. Consider the system

$$\dot{x} = y + x - x^3, \;\; \dot{y} = -y.$$

   Find the fixed points, classify them and construct the phase portrait.

2. (a) Consider the problem $u_{tt} = u_{xx}$ in the *infinite line* with the initial conditions: $u(x, 0) = 0$, $u_t(x, 0) = xe^{-x^2}$. Find the solution to this PDE at all times and sketch it at $t = 0$ and at a large time $t >> 0$.

   (b) Consider the problem $u_t + 3u = ku_{xx}$ in the *infinite line* with the initial condition $u(x, 0) = e^{-x^2}$. Find the solution to this PDE at all times and sketch it at $t = 0$ and at a large time $t >> 0$.

3. Consider the problem $u_t = u_{xx}$ with $u(0, t) = u(\pi, t) = 0$. Solve the PDE (by separating the variables and applying the boundary conditions) to obtain the most general possible solution satisfying these boundary conditions. Then, find the unique solution that satisfies

$$u(x, 0) = 3\sin(x) + 2\sin(4x)$$

4. Consider the ordinary differential equation

$$\dot{x} = x + \tanh(rx).$$

   Find the values of r at which the bifurcation occurs, classify the type of bifurcation and sketch the bifurcation diagram.

5. (a) Consider the eigenvalue problem

$$-f''(x) = \lambda f(x),$$

together with boundary conditions $f'(0) = a_0 f(0)$ and $f'(L) = -a_L f(L)$. Show that it cannot have negative eigenvalues when $a_0 > 0$ and $a_L > 0$.

(b) Consider the eigenvalue problem

$$f''''(x) = \lambda f(x),$$

together with boundary conditions $f(0) = f(L) = f''(0) = f''(L) = 0$. Again show that its eigenvalues λ cannot be negative.

6. Consider the system

$$\dot{x} = x - y - x(x^2 + 5y^2), \quad \dot{y} = x + y - y(x^2 + y^2).$$

(a) Classify the fixed point at the origin.

(b) Rewrite in polar coordinates, using $r\dot{r} = x\dot{x} + y\dot{y}$ and $\dot{\theta} = (x\dot{y} - y\dot{x})/r^2$.

(c) Determine a circle of maximum radius, r_1, centered on the origin such that all trajectories have a radially outward component on it.

(d) Determine a circle of minimum radius, r_2, centered on the origin such that all trajectories have a radially inward component on it.

(e) Assuming no other fixed points, prove that the system has a limit cycle somewhere in the trapping region $r_1 \leq r \leq r_2$.

7. (a) In the annular domain $a < r < b$, find the *radially symmetric* solution of the equation

$$\Delta u = 12$$

in 3 dimensions which satisfies *vanishing boundary conditions* at $r = a$ and $r = b$.

(b) Solve Laplace's Equation

$$\Delta u = 0$$

in the *exterior* of a disk ($r > a$) with boundary condition $u(a, \theta) = 3 + 2\cos(4\theta) - \sin(2\theta)$, and the condition that u must be bounded as $r \to \infty$.