BASIC EXAM – LINEAR ALGEBRA/ADVANCED CALCULUS
UNIVERSITY OF MASSACHUSETTS, AMHERST
DEPARTMENT OF MATHEMATICS AND STATISTICS
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Do 7 of the following 9 problems.

Passing Standard: For Master's level, 60% with three questions essentially complete (including at least one from each part). For Ph. D. level, 75% with two questions from each part essentially complete.

Show your work!

Part I. Linear Algebra

1. Find the dimension of the image and kernel of the matrix $A := \begin{pmatrix} 11 & 4 \\ 7 & 1 \\ 3 & 2 \\ 5 & 17 \\ 23 & 13 \end{pmatrix} \begin{pmatrix} 9 & 6 & 1 & 3 & 4 \\ 18 & 12 & 2 & 6 & 8 \end{pmatrix}$.

Also find a set of basis for each of these spaces.

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2. For each integer $n \geq 1$, determine the number of similarity classes of 2×2 matrix A with integer entries and of order exactly n, i.e. $A^n = I$ but $A^k \neq I$ for any integer $0 < k < n$. Show your work!

Note: Recall two real $n \times n$ matrices B, C are *similar* if $B = MCM^{-1}$ for some *real* invertible matrix M.

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3. For any two vectors $\vec{x}, \vec{y} \in \mathbf{R}^n$, denote by $\vec{x} \cdot \vec{y}$ the usual dot product (or inner product); it is a real number.

(a) For any linear transformation $T : \mathbf{R}^n \to \mathbf{R}^n$, show that there exists a unique linear transformation $T^* : \mathbf{R}^n \to \mathbf{R}^n$ that satisfies $(T\vec{x}) \cdot y = \vec{x} \cdot (T^*\vec{y})$ for all $\vec{x}, \vec{y} \in \mathbf{R}^n$.

(b) Let $A = (a_{ij})_{i,j}$ be the matrix for T with respect to the standard basis for $\mathbf{R}^n$. Write down the matrix for T^* with respect to the standard basis in terms of the a_{ij}.

Note: For both parts it is *not* enough to simply quote theorems or to write down the answers; you must justify your reasoning.

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4. Denote by $\phi : \mathbf{R}^3 \to \mathbf{R}^3$ the orthogonal projection map onto the plane P defined by $x - 2y - z = 0$. This is a linear transformation from $\mathbf{R}^3 \to \mathbf{R}^3$ determined by the conditions that (i) $\phi(\vec{x}) \in P$ for all $\vec{x} \in \mathbf{R}^3$, and (ii) for any $\vec{x} \in \mathbf{R}^3$, the length of $\phi(\vec{x}) - \vec{x}$ is equal to the distance between $\vec{x}$ and P.

Write down the matrix for ϕ with respect to the standard basis for $\mathbf{R}^3$. Justify your reasoning!

Part II. Advanced Calculus

1. Let $f(x), g(x)$ be real valued continuous functions on the interval closed $[a, b]$.
(a) Suppose that $g(x) \geq 0$ on $[a, b]$. Show that there exists a number $\xi \in [a, b]$ such that

$$\int_a^b f(x)\ g(x)\ dx = f(\xi) \int_a^b g(x) dx.$$

(b) Give an example to show that the equality above is false if we do *not* require that $g(x) \geq 0$ on $[a, b]$.

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2. A rectangle with length L and width W is cut into four smaller rectangles by two lines parallel to the sides. Find the maximum and minimum of the sum of the squares of the areas of the smaller rectangles.

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3. Denote by $\mathcal{C}$ the set of all continuous functions on $[0, 1]$. For any two functions $f, g \in \mathcal{C}$, we have the Cauchy-Schwartz inequality

$$\int_0^1 |f(x)g(x)| dx \leq \Big(\int_0^1 |f(x)|^2 dx\Big)^{1/2} \Big(\int_0^1 |g(x)|^2 dx\Big)^{1/2}.$$

Determine all functions $f, g \in \mathcal{C}$ for which this is an *equality*. Justify your reasoning.

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4. Let $a_1, a_2, \ldots$ be a sequence of real numbers that converges to A, and let $b_1, b_2, \ldots$ be a sequence of real numbers that converges to B. Does the limit $\lim_{n\to\infty} \dfrac{a_1 b_1 + \ldots + a_n b_n}{n}$ exist? Find the limit if so, and give a counter-example if not. Justify your reasoning!

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5. For any real number x, denote by $[\![x]\!]$ the largest integer $\leq x$. Compute the double integral $\iint_R [\![x + y]\!] dA$, where $R = \{(x, y) \in \mathbf{R}^2 :\ 1 \leq x \leq 3,\ 2 \leq y \leq 5\}$.
