DEPARTMENT OF MATHEMATICS AND STATISTICS UMASS - AMHERST BASIC EXAM - PROBABILITY WINTER 2011

Work all problems. 60 points are needed to pass at the Masters Level and 75 to pass at the Ph.D. level. Each question is worth 20 points.

1. Let A be a bounded region in $\mathbb{R}^2$ and $|A|$ be its area. The boundary of A is known and, for any $(x, y) \in \mathbb{R}^2$ it is easy to determine whether $(x, y) \in A$. However, $|A|$ is not known and cannot be calculated analytically. The following method is proposed to estimate $|A|$:

   (a) Construct a rectangle B that contains A.

   (b) Generate N (a large integer) points at random, uniformly, in B.

   (c) Let X be the number of generated points that lie within A.

   (d) Use X/N as an estimate of $|A|/|B|$ and $|B| \times (X/N)$ as an estimate of $|A|$.

   The user can choose B, as long as it's big enough to contain A. If the goal is to estimate $|A|$ as accurately as possible, what advice would you give the user for choosing the size of B? Should $|B|$ be large, small, or somewhere in between? Justify your answer. *Hint: think about Binomial distributions.*

2. You go to the bus stop to catch a bus. You know that buses arrive every 15 minutes, but you don't know when the next is due. Let T be the time elapsed, in hours, since the previous bus. Adopt the prior distribution $T \sim \text{Unif}(0, 1/4)$.

   (a) Find $\text{E}[T]$.

   Passengers, apart from yourself, arrive at the bus stop according to a Poisson process with rate $\lambda = 2$ people per hour; i.e., in any interval of length ℓ, the number of arrivals has a Poisson distribution with parameter 2ℓ and, if two intervals are disjoint, then their numbers of arrivals are independent. Let X be the number of passengers, other than yourself, waiting at the bus stop when you arrive.

   (b) Suppose $X = 1$. Write an intuitive argument for whether that should increase or decrease your expected value for T. I.e., is $\text{E}[T|X = 1]$ greater than, less than, or the same as $\text{E}[T]$?

   (c) Find the density of T given $X = 1$, up to a constant of proportionality. It is a truncated version of a familiar density. What is the familiar density?

3. A discrete-time Markov chain is a series of indexed random variables, $\{X_0, X_1, X_2, \ldots\}$ which displays the Markov property, namely

$$\Pr(X_{n+1} = j | X_0 = x_0, X_1 = x_1, X_2 = x_2, \ldots X_n = i) = \Pr(X_{n+1} = j | X_n = i).$$

Consider such a Markov chain in which there are only finitely many possible x's and in which the so-called transition probabilities are given by the matrix $\mathbf{p}$ such that

$$\mathbf{p}_{ij} = \Pr(X_{n+1} = j | X_n = i),$$

constant for all $n \geq 0$.

(a) Give an expression in terms of $\mathbf{p}$ for the probability that $X_{n+2} = j$ given $X_n = i$.

(b) Give an expression in terms of $\mathbf{p}$ for the probability that $X_{n+m} = j$ given $X_n = i$. For full credit, whenever possible, express your answer using matrix notation rather than functions of the matrix elements.

(c) Prove that for any Markov chain, $\Pr(X_3 = x_3 | X_0 = x_0, X_1 = x_1) = \Pr(X_3 = x_3 | X_1 = x_1)$.

4. Suppose X_1 and X_2 are random variables with joint density function $f(x_1, x_2) = c$ when $x_1 + x_2 \leq 1$ and both x_1 and x_2 are non-negative. The density $f(x_1, x_2) = 0$ otherwise. Except for part (a), purely graphical solutions will not get full credit.

   (a) Draw a picture to show the x_1 and x_2 values where the density is non-zero.

   (b) What is c?

   (c) What is the probability that $X_1 > X_2$?

   (d) Are X_1 and X_2 independent? Why or why not?

   (e) What is the density of $Y = 1/X_1$?

5. Suppose X_1 and X_2 are independent and identically distributed random variables with density $f(x) = \lambda \exp(-\lambda x), x \geq 0$, and $f(x) = 0$ otherwise.

   (a) The moment generating function of a random variable X is $M_X(t) = \mathrm{E}[e^{tX}]$. Find the moment generating function of X_1.

   (b) Use the moment generating function to show that $Y = X_1 + X_2$ has density $f(y) = \lambda^2 y \exp(-\lambda y), y \geq 0$, and $f(y) = 0$ otherwise.

   (c) Suppose $\lambda = 1$. Let $c > 0$. Show that the density of $X_1 | X_1 > c$ is $\exp(-x) / \{1 - \exp(-c)\}, x > c$ and 0 otherwise.

   (d) Suppose $\lambda = 1$. Let $c > 0$. Find the $E(X_1 | X_1 > c)$.