Department of Mathematics and Statistics
Basic Probability Exam
January 2010

Work all problems. Show your work; explain your answers; state theorems used whenever possible.

1. A gene has two possible forms (alleles): A and a. Thus there are three possible genotypes: AA, aA, and aa. Number them 1, 2, and 3, respectively. Assume that their proportions in the population are p^2, $2pq$, and q^2, respectively ($q = 1 - p$).

   For a family with a father, a mother, and one child, let the random variables F, M, and C denote the genotypes of the father, mother, and child, respectively. For example, F is either 1, 2, or 3, according to the genotype of the father. Assume that F is independent of M, i.e. that the population mates randomly, and that the conditional distribution of C given (F, M) is determined by the familiar rules of genetics. (Children inherit one gene from each parent; each parent's gene has probability 0.5 of being chosen; the mother's contribution is independent of the father's contribution.) Let $p_{ik} = \Pr[C = k \mid M = i]$, the conditional probability that the child is of type k given that the mother (or father) is of type i. Compute the nine probabilities p_{ik} in terms of p and q.

2. Let $\vec{Y}$ have a trivariate Gaussian distribution with mean vector $\vec{\mu}$ and covariance matrix Σ, where

$$\vec{Y} = \begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \end{pmatrix}, \quad \vec{\mu} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}, \quad \Sigma = \begin{pmatrix} 1 & \rho & 0 \\ \rho & 1 & \rho \\ 0 & \rho & 1 \end{pmatrix}.$$

   (a) For which values of ρ are $Y_1 + Y_2 + Y_3$ and $Y_1 - Y_2 - Y_3$ statistically independent?

   (b) What is the distribution of $Y_1 + Y_2 + Y_3$, including its name and associated parameters.

3. Suppose that X is a random variable with density $(3x + 1)/8$ on the interval $(0, 2)$. Let Y be the area of a circle of radius X. Find the density of Y.

4. (a) A continuous random variable Y takes values on the interval $(0, \infty)$. Show $\mathrm{E}[Y] = \int_0^\infty \Pr[Y \geq y]\, dy$. Hint: you may use the fact that $y = \int_0^y dz$.

   (b) A discrete random variable X takes values on the positive integers $1, 2, \ldots$. Show $E[X] = \sum_{x=1}^\infty \Pr[X \geq x]$.

5. A family of densities is called a *univariate natural exponential* family if, for some function $A(\theta)$, the density of X given θ can be expressed as

$$p(x \mid \theta) = h(x) e^{\theta x - A(\theta)}.$$

   Suppose that X has such a density.

   (a) Show that the moment generating function $M_{X \mid \theta}(t) = \mathrm{E}[e^{tX}]$ is $e^{[A(\theta+t) - A(\theta)]}$.

   (b) Show that $\mathrm{E}[X] = A'(\theta)$.