University of Massachusetts
Department of Mathematics and Statistics
Advanced Exam in Geometry
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Do 5 out of the following 7 problems. Indicate clearly which questions you want graded. *Passing standard:* 70% with three problems essentially complete. **Justify all your answers.**

1. a) Show that
$$M = \{([x_0 : x_1 : x_2], t) \in \mathbb{P}^2 \times \mathbb{R} : x_0 + x_1 t + x_2 t^2 = 0\}$$
is an embedded submanifold of $\mathbb{P}^2 \times \mathbb{R}$, where $\mathbb{P}^2$ denotes the real projective plane, i.e. the space of lines through the origin in $\mathbb{R}^3$.
b) Let $\pi\colon M \to \mathbb{P}^2$ be the restriction to M of the projection from $\mathbb{P}^2 \times \mathbb{R}$ onto the first factor. Find the regular values of π.

2. Let $E = \{(x, \ell) | x \in \mathbb{R}^3, \ell \in \mathbb{P}^2$, such that $x \in \ell\} \subset \mathbb{R}^3 \times \mathbb{P}^2$, and let $\pi : E \to \mathbb{P}^2$ be the map sending (x, ℓ) to ℓ. (As usual, $\mathbb{P}^2$ is the space of lines in $\mathbb{R}^3$ passing through the origin.)
a) Show that E is a smooth vector bundle over $\mathbb{P}^2$ of rank 1.
b) Is E a trivial bundle? Explain why.

3. Let X be a smooth vector field on M.
a) Prove that if $X(p) \neq 0$ then there exist local coordinates $(u_1, \dots, u_n)$ in a neighborhood U of p such that $X|_U = \partial/\partial u_1$.
b) Consider the vector field in $\mathbb{R}^2$:
$$X = y\,\frac{\partial}{\partial x} - x\,\frac{\partial}{\partial y}.$$
and let $p = (1, 0)$. Exhibit local coordinates (u_1, u_2) in a neighborhood U of p such that $u_1(p) = u_2(p) = 0$ and $X|_U = \partial/\partial u_1$.

4. Let G be a finite group acting smoothly and freely on a smooth manifold M, and let $N = M/G$ be the quotient manifold. Denote by $\pi : M \to N$ the quotient map. For $0 \leq k \leq \dim M$, let
$$\pi^* : H^k_{dR}(N) \to H^k_{dR}(M)$$
be the induced homomorphism on the de Rham cohomology. Show that:
a) π^* is injective;
b) The image of π^* is $H^k_{dR}(M)^G := \{\alpha \in H^k_{dR}(M) | g^*\alpha = \alpha, \ \forall g \in G\}$.

5. Let (M, ω) be a symplectic manifold with $\pi_1(M) = 1$, and let $\psi_t : M \to M$ be a flow generated by a smooth vector field X such that $\psi_t^* \omega = \omega$.
   a) Show that there is a smooth function $H : M \to \mathbb{R}$ such that $i(X)\omega = dH$, where $i(X)\omega(Y) := \omega(X, Y)$, for every vector field Y on M.
   b) Consider the special case where $M = S^2 \subset \mathbb{R}^3$ is the unit sphere and ω is the induced area form. If ψ_t is the flow given by rotation of angle t about the z-axis, i.e.,

$$\psi_t(x, y, z) = (x \cos t - y \sin t, x \sin t + y \cos t, z),$$

   compute the function H in part (1) (note that H is only determined up to a constant).

6. Let $f \in C^\infty(\mathbb{R})$ be such that $f(x) > 0$ for all $x \in \mathbb{R}$. Consider the Riemannian metric in $\mathbb{R}^2$ given by:

$$g = dx^2 + f^2(x)\, dy^2.$$

   a) Find all functions $f(x)$ for which the Gaussian curvature K of $(\mathbb{R}^2, g)$ is constant.
   b) Find $\operatorname{grad}(U)$, where $U \in C^\infty(\mathbb{R}^2)$.
   c) Find $\operatorname{div}(X)$, where X is a smooth vector field in $\mathbb{R}^2$.

7. Let (M, g) be a Riemannian manifold and ∇ be the Levi-Civita connection. Let $\{E_i\}$ be a local orthonormal frame and $\{\phi^i\}$ be the dual frame.
   a) Recall that the connection 1-forms $\{\omega_i^j\}$ of ∇ are defined by

$$\nabla_X E_i = \sum_j \omega_i^j(X) E_j.$$

   Deduce Cartan's first structure equation:

$$d\phi^j = \sum_i \phi^i \wedge \omega_i^j.$$

   b) Recall the curvature 2-forms $\{\Omega_i^j\}$ are defined by

$$R(X, Y)E_i = \sum_j \Omega_i^j(X, Y) E_j,$$

   where $R(X, Y)Z := \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$. Deduce Cartan's second structure equation:

$$\Omega_i^j = d\omega_i^j - \sum_k \omega_i^k \wedge \omega_k^j.$$