DEPARTMENT OF MATHEMATICS AND STATISTICS
UNIVERSITY OF MASSACHUSETTS
BASIC EXAM - STATISTICS
Friday, January 23, 2009

Work all five problems. Sixty points are needed to pass at the Master's level and seventy-five at the Ph.D. level.

1. (**30** points) Let $f(\vec{x};\theta)$ be the pdf of $\vec{X}$ associated with parameter θ. Let $T = t(\vec{X})$ be a statistic. Give a precise **definition** or **statement** for the following:

   (a) that T is a sufficient statistic.

   (b) that T is a minimal sufficient statistic.

   (c) that T is a complete sufficient statistic.

   (d) the Lehmann-Scheffe Theorem relating to UMVUE.

   (e) that $T > c$ is a size α Uniformly Most Powerful (UMP) test for testing $H_0 : \theta \leq \theta_0$ versus $H_1 : \theta > \theta_0$.

   (f) The Neyman-Pearson fundamental lemma for testing $H_0 : \theta = \theta_0$ versus $H_1 : \theta = \theta_1$, where θ_0 and θ_1 are two different given numbers.

2. (**20** points) Given n independent pairs (X_i, Y_i), each with joint pdf:

$$f(x, y|\theta) = e^{-\theta x - y/\theta}, \quad \text{for } x > 0,\ y > 0,\ \theta > 0$$

   (a) Find the MLE of θ. You need to justify why it is an MLE.

   (b) It is known that $(\sum_{i=1}^n X_i, \sum_{i=1}^n Y_i)$ is the minimal sufficient statistic for θ. Can the MLE be sufficient? Explain why.

3. (**25** points) Let $X_1, \ldots X_n$ be a random sample from Bernoulli(p) with probability function

$$f(x|p) = p^x(1-p)^{1-x}, \text{ for } x = 0, 1;\ 0 < p < 1.$$

   (a) Find the MLE of p **and** $\tau = p(1-p)$, respectively.

   (b) Use the delta method to find an approximate 95% confidence interval for τ.

   (c) Consider the prior pdf for the parameter p as $Beta(r, s)$:

$$g(p|r, s) = \frac{\Gamma(r+s)}{\Gamma(r)\Gamma(s)} p^{r-1}(1-p)^{s-1}, \text{ for } r > 0, s > 0.$$

      i. Find the posterior distribution of p.

      ii. Find the Bayes estimate of p under squared error loss. You may use the fact that for $U \sim Beta(\alpha, \beta)$,

$$E(U) = \frac{\alpha}{\alpha+\beta},\quad Var(U) = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}, \text{ for } \alpha > 0, \beta > 0,$$

      and $\Gamma(k) = (k-1)\Gamma(k-1)$.

      iii. Is the Bayes estimator consistent? Justify your answer.

4. (**25** points) Let $X_1, \ldots X_n$ be a random sample from a $N(\mu, 1)$ distribution, where the mean μ is unknown, which has pdf:

$$f(x|\mu) = \frac{1}{\sqrt{2\pi}} \exp\{-\frac{1}{2}(x-\mu)^2\}, \text{ for } -\infty < x < \infty, -\infty < \mu < \infty.$$

   (a) Consider testing

   $$H_0: \ \mu = 0 \ \text{ versus } H_1: \ \mu = \mu_1,$$

   where $\mu_1 > 0$ is a given number.

      i. Derive the most powerful test (MPT). You need to specify the critical region for the test of size α, with critical value being a quantile of a standard normal distribution (with justifications).

      ii. Find the power of the MPT when $\mu = 1/2$ for $n = 16$ and $\alpha = 0.05$. (A table of the $N(0,1)$ distribution is enclosed.)

   (b) Now consider another problem of testing

   $$H_0: \ \mu = 0 \ \text{ versus } H_1: \ \mu \neq 0.$$

      i. Derive the likelihood ratio test (LRT). You need to specify the critical region for the test of size α, with critical value also being a quantile of the standard normal distribution (with justifications).

      ii. Find the power of the LRT when $\mu = 1/2$ for $n = 16$ and $\alpha = 0.05$.

   (c) Based on the power computations, can the above LRT be a UMP test? Explain why.