University of Massachusetts
Department of Mathematics and Statistics
Advanced Exam in Geometry
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Do 5 out of the following 8 problems. Indicate clearly which questions you want graded. *Passing standard:* 70% with three problems essentially complete. **Justify all your answers.**

1. Let $F, G : \mathbb{R}^{n+1} \to \mathbb{R}$ be C^∞ maps and suppose that $0 \in \mathbb{R}$ is a regular value of F. Set $X := F^{-1}(0)$ and $g := G|_X : X \to \mathbb{R}$. Prove that a point $p \in X$ is a critical point of g if and only if there exists $\lambda \in \mathbb{R}$ such that

$$dF_p \;=\; \lambda\, dG_p.$$

2. Let M be a 3-dimensional smooth manifold and let α be a nowhere zero smooth 1-form on M. Define: $\Delta(p) \;:=\; \{v \in T_p(M) \mid \alpha(p)(v) = 0\}$.

   (a) Prove that Δ is a smooth 2-dimensional distribution in M.

   (b) Prove that Δ is integrable (involutive) if and only if

$$\alpha \wedge d\alpha = 0.$$

3. Prove or disprove the following statements:

   (a) If a 1-form α on a smooth manifold M is nowhere zero and θ is another 1-form such that $\theta \wedge \alpha = 0$, then there exists $f \in C^\infty(M)$ such that

$$\theta \;=\; f\,\alpha.$$

   (b) If a 1-form α on $M = \mathbb{R}^2 \backslash \{0\}$ satisfies $d\alpha = 0$, then there exists $f \in C^\infty(\mathbb{R}^2 \backslash \{0\})$ such that $\alpha = df$.

4. Let M be an n-dimensional smooth manifold, and let $S \subset M$ be an embedded submanifold of dimension $n-1$. Show that the normal bundle $N_S M := (TM|_S)/TS$ is trivial if and only if there exists an open neighborhood $U \subset M$ of S and a smooth function $f \in C^\infty(U)$ so that 0 is a regular value of f and $S = f^{-1}(0)$.

5. A manifold M is called *symplectic* if there exists a 2-form $\omega \in \Lambda^2(M)$ which satisfies (1) $d\omega = 0$ and (2) ω_p is a nondegenerate bilinear form on T_pM for all $p \in M$. Prove that

   (a) any orientable, 2-dimensional manifold is symplectic, and

   (b) For any $n \geq 2$, the sphere S^{2n} is not symplectic but the torus $T^{2n} = (S^1)^{2n}$ is.

6. Let (M, g) be a connected, orientable Riemannian manifold.

   (a) Define the *divergence*, $\operatorname{div}_g(X)$, of a vector field X on M.

   (b) Prove that $\operatorname{div}_g(X)$ is independent of the choice of orientation.

   (c) Let $\lambda \in C^\infty(M)$ and let g_λ the metric, conformal to g, defined by

   $$g_\lambda(X, Y) \;=\; e^{2\lambda}\, g(X, Y).$$

   Prove that

   $$\operatorname{div}_{g_\lambda}(X) \;=\; \operatorname{div}_g(X) + (\dim M)\, X(\lambda).$$

7. Let $M = \{(x, y) \in \mathbb{R}^2 : y > 0\}$, with the metric $g = y\, dx^2 + dy^2$, i.e.

   $$g(\partial/\partial x, \partial/\partial x) = y \;; \quad g(\partial/\partial y, \partial/\partial y) = 1 \;; \quad g(\partial/\partial x, \partial/\partial y) = 0.$$

   (a) Compute the Gaussian curvature of (M, g).

   (b) Given that the Christoffel symbols of g relative to the frame $\partial/\partial x$, $\partial/\partial y$ are given by:

   $$\Gamma^1_{11} = \Gamma^2_{12} = \Gamma^1_{22} = \Gamma^2_{22} = 0 \;; \quad \Gamma^2_{11} = -1/2 \;; \quad \Gamma^1_{12} = 1/(2y)\,,$$

   write the differential equations for a geodesic in (M, g).

   (c) Determine whether vertical or horizontal lines are geodesics and, if so, what is the appropriate parametrization.

8. Suppose that a Lie group G acts on a smooth manifold M; i.e., there is a smooth map $a\colon G \times M \to M$ satisfying

   $$a(g, a(h, p)) = a(gh, p)$$

   for all $g, h \in G$ and $p \in M$.

   (a) Show that for any $p \in M$, the stabilizer

   $$G_p = \{g \in G \mid a(g, p) = p\}$$

   is a Lie subgroup of G.

   (b) Show that there is a unique linear transformation $X \mapsto \widetilde{X}$ from the Lie algebra $\operatorname{Lie} G$ to the space of smooth vector fields on M such that

   $$\widetilde{X}_{a(g,p)} = (\phi_p)_*(X_g)$$

   for all $p \in M$, $g \in G$, where $\phi_p\colon G \to M$ is given by $\phi_p(g) = a(g, p)$.

   (c) Show that $\operatorname{Lie} G_p = \{X \in \operatorname{Lie} G \mid \widetilde{X}_p = 0\}$