DEPARTMENT OF MATHEMATICS AND STATISTICS
UNIVERSITY OF MASSACHUSETTS
BASIC EXAM – PROBABILITY
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Work all problems. 60 points are needed to pass at the Master's level and 75 to pass at the Ph.D. level.

1. (25 points) Suppose that X has density function $f(x) = b(1 - x^2), |x| < c$, and $f(x)$ is zero otherwise.

   (a) Find constants b and c so that $f(x)$ is a density function. (Note that there is more than one right answer.)

   (b) Derive the first and second moments of X.

   (c) Derive the conditional density of X given that X is greater than 0.

2. (15 points) Suppose we have three cards. The first one is blank on both sides, the second has an X on one side and is blank on the other, and the third has an X on both sides. We run an "experiment" where we choose one card at random and then look at one side of the chosen card at random.

   (a) What is the probability that you see an X?

   (b) What is the probability that you see an X and the other side has an X too?

   (c) Suppose we run the experiment above and we see an X. Given that outcome, what is the probability that the other side of the card has an X on it too?

3. (20 points) Let X be a random variable with $E(X) = \mu$ and $\Pr(X = \mu) < 1$.

   (a) Is $E\{\exp(X)\}$ equal to, less than, or greater than $\exp(EX) = \exp(\mu)$ in general? Why?

   (b) Now, suppose $X \sim N(0, 1)$ with density $\frac{1}{\sqrt{2\pi}} \exp(-x^2/2)$.

      i. What is the moment generating function of X?

      ii. Use the previous result (part i) to derive $E\{\exp(X)\}$ in this case.

4. (25 points) Let $Y_1, \ldots, Y_n$ be a random sample from a Poisson distribution with rate λ and $Pr(Y_i = k) = \exp(-\lambda)\lambda^k/k!, k = 0, 1, \ldots$. Let $\overline{Y}_n = n^{-1} \sum_{i=1}^n Y_i$.

   (a) State the central limit theorem in general, and then use it to argue that $n^{1/2}(\overline{Y}_n - \lambda)/\sqrt{\lambda}$ converges in distribution to a standard normal.

   (b) Let $Z_i = 1$ if $Y_i > 0$ and $Z_i = 0$ otherwise. Let $\widehat{\alpha}_n = n^{-1} \sum_{i=1}^n Z_i$. Find a μ and σ so that $n^{1/2}(\widehat{\alpha}_n - \mu)/\sigma$ converges in distribution to a standard normal.

   (c) Given that $\overline{Y}_n$ converges to λ in probability as n goes to infinity, define a function of $Y_1, \ldots, Y_n$ that converges in probability to σ in the previous question as n goes to infinity. Explain your answer, and name the results that you use to show convergence in probability.

5. (15 points) Let X have an exponential distribution with CDF $1 - exp(-x/\lambda), x \geq 0, \lambda > 0$.

(a) What is the pdf of X?

(b) Suppose that Y is independent of X and has the same distribution. Show that the distribution of $Z = X + Y$ is *gamma*$(2, \lambda)$ with density $f(z) = \frac{1}{\Gamma(2)\lambda^2} z \exp(-z/\lambda)$.