DEPARTMENT OF MATHEMATICS & STATISTICS
BASIC EXAM: NUMERICS
JANUARY 2007

Do five of the following problems. All problems carry equal weight.
Passing Level:
Masters: 60% with at least two substantially correct.
PhD: 75% with at least three substantially correct.

1. Suppose that we wish to approximate the integral

$$\int_1^2 f(x)dx$$

by a formula of the form

$$af(0) + bf(1).$$

Note that this formula uses $f(0)$ and $f(1)$ while the integral is over the interval $[1, 2]$. Find the most accurate method of this form along with the error term.

2. Suppose A is a real $n \times n$ symmetric matrix, i.e. $A \in \mathbb{R}^{n\times n}$ and $A^T = A$.

   (a) Define what is means for A to be positive definite.

   (b) Suppose A is positive definite. Prove that the diagonal entries of A are all positive.

   (c) Suppose A is positive definite. Prove that the largest entry of A in absolute value lies on the diagonal.

3. Consider the *natural* cubic spline function $s(x)$ defined on $[0, 2]$ which interpolates the function $f(x)$ using the following data:

x_i	0	1	2
$f(x_i)$	f_0	f_1	f_2

Therefore, $s(x)$ is defined piecewise by 2 cubic polynomials,

$$s(x) = \begin{cases} s_0(x) & 0 \le x \le 1, \\ s_1(x) & 1 \le x \le 2. \end{cases}$$

Suppose that $s_0(x) = 1 + 2x + 4x^3$. Then

   (a) What are the values of f_0 and f_1?

   (b) What is the value of f_2?

   (c) Suppse further that $f \in C^2[0, 2]$. Show

$$\int_0^2 [s''(x)]^2 \, dx \le \int_0^2 [f''(x)]^2 \, dx.$$

4. (a) Consider the iteration $x_{n+1} = x_n^3$, Give a detailed discussion of the behavior of the sequence $\{x_n\}$ in dependence of x_0. In particular, discuss the fixed points, convergence and the order of convergence.

   (b) Same for $x_{n+1} = x_n^{\frac{1}{3}}$.

5. For solving $y' = f(x, y)$, consider the numerical method

$$y_{n+1} = y_n + \frac{h}{2}(y'_n + y'_{n+1}) + \frac{h^2}{12}(y''_n - y''_{n+1}),$$

where $n = 0, 1, ...,$ and $h = x_{n+1} - x_n$(the step size).

   (a) Show that this method is at least fourth order accurate.

   (b) For the equation $y' = \lambda y$, $y(0) = \epsilon \neq 0$, show that the method will not blow up if λ is negative and real as $n \to \infty$.

6. Consider the following matrix.

$$\mathrm{A} = \begin{pmatrix} 4 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 4 \end{pmatrix}.$$

   (a) Find the LU decomposition of A.

   (b) Use the LU factors to solve the system Ax=b, where $b = (-3, 4, 5)^T$.

7. Suppose $f(\alpha) = 0$ and $f'(\alpha) \neq 0$, i.e., α is a *simple* root of $f(x)$. Then the convergence rate of Newton's method

$$x_{n+1} = g(x_n) = x_n - \frac{f(x_n)}{f'(x_n)},$$

is *at least* second order if x_0 is sufficiently close to α.

Suppose now that α is a *multiple* root of $f(x)$ of multiplicity $p \geq 2$,

$$f(\alpha) = f'(\alpha) = f''(\alpha) = ... = f^{(p-1)}(\alpha) = 0, \text{ and } f^{(p)}(\alpha) \neq 0.$$

In this case, we can write

$$f(x) = (x - \alpha)^p h(x)$$

for some function $h(x)$, and $h(\alpha) \neq 0$.

   (a) Suppose α is a root of $f(x)$ with multiplicity $p \geq 2$. Write out the iteration function $g(x)$ for Newton's method. (Note: it will involve $h(x)$ and $h'(x)$).

   (b) Show that the convergence rate of Newton's method in the case of a multiple root is only linear, with rate $1 - 1/p$.