

University of Massachusetts
Department of Mathematics and Statistics
Advanced Exam in Geometry
January 2007

Do 5 out of the following 7 problems. Indicate clearly which questions you want graded. *Passing standard:* 70% with three problems essentially complete. **Justify all your answers.**

Problem 1. Give examples for the following maps between a pair of compact manifolds M, N , or prove none exists:

1. An injective immersion $f: M \rightarrow N$ that is not surjective.
2. A surjective submersion $f: M \rightarrow N$ that is not injective.
3. A surjective immersion $f: M \rightarrow N$ that is not injective.

It is not necessary to give explicit formulas in coordinates for your maps, and the manifolds need not be the same in the different examples.

Problem 2. Consider the triply-periodic 1-form

$$\alpha = \sin^{2007} x \, dx + dy + \cos^{2007} z \, dz$$

on \mathbb{R}^3 .

1. Show there is a unique $h: \mathbb{R}^3 \rightarrow \mathbb{R}$ such that $dh = \alpha$ and $h(0, 0, 0) = 0$.
2. Regarding α as a 1-form on the torus $T^3 = \mathbb{R}^3/2\pi\mathbb{Z}^3$, determine whether or not there is a $g: T^3 \rightarrow \mathbb{R}$ such that $dg = \alpha$. (Hint: Compute the deRham class $[\alpha] \in H^1(T^3, \mathbb{R})$.)

Problem 3. A manifold is called symplectic if there exists a 2-form which is closed and non-degenerate. Prove that

1. A symplectic manifold is orientable and even-dimensional.
2. Any orientable, 2-dimensional manifold is symplectic.
3. If M, N are symplectic, so is $M \times N$.

4. For any $n \geq 2$, the sphere S^{2n} is orientable, even-dimensional, but not symplectic.

Problem 4. Show that every real line bundle over S^n is trivial if $n \neq 1$, that every complex line bundle over S^n is trivial when $n \neq 2$, and give examples showing that these dimension restrictions are necessary.

Problem 5. Let $G \subset SL(2, \mathbb{C})$ be the set of all 2-by-2 complex matrices A with determinant 1 such that $A * QA = Q$, where $A*$ is the conjugate transpose of A and Q is the diagonal matrix with entries 1 and -1 .

1. Prove that G is a Lie group, calculate its Lie algebra and compute its dimension.
2. Determine a maximal abelian subalgebra of its Lie algebra and a corresponding maximal torus in G .

Problem 6. Prove that any smooth manifold equipped with a smooth finite group action admits a Riemannian metric which is invariant under the group action. (Hint: first show that any positive linear combination of Riemannian metrics is again a Riemannian metric.)

Problem 7. The graph $z = f(x, y)$ of the function $f(x, y) = x^2 - y^2$ defines a smooth surface $\Sigma \subset \mathbb{R}^3$ in Euclidean space.

1. Determine the induced Riemannian metric on Σ and show that it is complete.
2. Find the geodesics through the origin in the directions $(1, 0)$, $(0, 1)$ and $(1, 1)$.
3. Compute the Gauss and mean curvatures of Σ at the origin.