DEPARTMENT OF MATHEMATICS AND STATISTICS
UNIVERSITY OF MASSACHUSETTS
BASIC EXAM: PROBABILITY
JANUARY 2005

Work all problems. Sixty points are needed to pass at the Master's level and seventy-five at the Ph.D. level

1. (20 points) A Poisson random variable with mean μ has pmf:

$$f(x) = \frac{\exp(-\mu)\mu^x}{x!}, x = 0, 1, 2, \dots$$

   (a) Let X be Poisson with mean μ. Compute the moment generating function of X. It may help to remember that:

$$\exp(y) = \sum_{k=0}^{\infty} \frac{y^k}{k!}.$$

   (b) Let X_1, X_2 be independent Poisson variables with means μ_1, μ_2, and let a_1, a_2 be positive constants. What is the moment generating function of $Y = \sum_{i=1}^{2} a_i X_i$?

   (c) What is the distribution of Y?

2. (20 points) Let X and Y have the joint density function $f(x, y) = c, 0 \leq x \leq y \leq 1$.

   (a) Find c.

   (b) What is the marginal pdf of X?

   (c) Are X and Y independant? Why or why not.

3. (20 points) A weed is exposed to a known dose of weed killer (X). The weed either survives ($Y = 1$) or dies ($Y = 0$). Suppose the weed has an unobserved natural tolerance to the weed killer (denoted by Z), and assume that this tolerance has a standard normal distribution. Further, suppose that the weed survives if an only if $Z > -X$. Note that Z is random and X is fixed.

   (a) What is the probability that the weed survives?

   (b) What is the distribution of Z given that the weed is not killed?

   (c) Derive the moment generating function for Z given that $Y = 1$. You may express your answer as an unsimplified integral that involves the standard normal pdf ($\phi(\cdot)$), cdf ($\Phi(\cdot)$), and other functions.

   (d) Use the result from the previous part to derive:

$$E(Z|Y = 1) = \frac{\phi(-X)}{1 - \Phi(-X)} = \frac{\phi(X)}{\Phi(X)}.$$

4. (20 points) A game is played with n coins. Coins 1 through $n-1$ are "fair" and land heads with probability $1/2$. The nth coin has two heads; it always lands heads up. The game consists of drawing coins blindly from the bag, flipping them, and replacing them back into the bag.

   (a) Let T be the number of coins that must be drawn and flipped until one sees a total of 3 tails. What is the mean of T?

   (b) What is the probability that T strictly exceeds 6?

   (c) Suppose one coin is drawn from the bag, flipped, and it lands heads. What is the probability that it is the unfair coin (the nth coin)?

5. (20 points) Joe walks to and from work each day. The commute to work, T_i, has mean μ_T and variance $\sigma_T 2$. The commute from work, F_i, has mean μ_F and variance $\sigma_F 2$. Further, suppose T_i and F_i are mutually independent. Let $D_i = T_i - F_i$.

   (a) What are the mean and variance of D_i?

   (b) Let $\overline{D}_{100}$ be the mean difference over 100 days: $\overline{D}_{100} = \sum_{i=1}^{100} D_i/100$. Write an approximation for the probability that $\overline{D}_{100}$ is negative.