BASIC EXAM: NUMERICS

Do five of the following problems. All problems carry equal weight.
Passing Level:
Masters: 60% with at least two substantially correct.
PhD: 75% with at least three substantially correct.

1. Suppose $f(\alpha)=0$ and $f'(\alpha) \neq 0$, i.e., α is a *simple* root of $f(x)$. Then the convergence rate of Newton's method

$$x_{n+1} = g(x_n) = x_n - \frac{f(x_n)}{f'(x_n)},$$

is *at least* second order if x_0 is sufficiently close to α. Suppose now that α is a *multiple* root of $f(x)$ of multiplicity $p \geq 2$,

$$f(\alpha) = f'(\alpha) = f''(\alpha) = ... = f^{(p-1)}(\alpha) = 0, \text{ and } f^{(p)}(\alpha) \neq 0.$$

In this case, we can write

$$f(x) = (x - \alpha)^p h(x)$$

for some function $h(x)$, and $h(\alpha) \neq 0$.

a) Suppose α is a root of $f(x)$ with multiplicity $p \geq 2$. Write out the iteration function $g(x)$ for Newton's method. (Note: it will involve $h(x)$ and $h'(x)$).

b) Show that the convergence rate of Newton's method in the case of a multiple root is only **linear**, with rate $1 - 1/p$.

2. The Chebyshev polynomials are defined for $x \in [-1, 1]$ by $\mathrm{T}_n(x) = \cos(n\theta)$, $x = \cos\theta$.

a) Derive the 3-term recurrence relation,

$$\mathrm{T}_{n+1}(x) = 2x\mathrm{T}_n(x) - \mathrm{T}_{n-1}(x) \ .$$

(Hint: One approach is to write out the lefthand side using the definition of T_{n+1} and applying the trig identity

$$\cos(A + B) = \cos A \cos B - \sin A \sin B$$

when appropriate.)

b) Given $\mathrm{T}_0(x) = 1$ and $\mathrm{T}_1(x) = x$, use the recurrence relation to find $\mathrm{T}_2(x)$ and $\mathrm{T}_3(x)$.

c) Find the roots of $T_2(x)$, and denote them by x_0 and x_1. Determine coefficients c_0 and c_1 such that the Gaussian quadrature approximation

$$\int_{-1}^{1} \frac{f(x)}{\sqrt{1 - x^2}} \, dx \approx c_0 f(x_0) + c_1 f(x_1)$$

is **exact** if f is any polynomial of degree 3 or less. (Note: Recall that $\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1}(x) + C$.)

3. Consider the symmetric real matrix

$$A = \begin{pmatrix} -4 & 1 \\ 1 & -4 \end{pmatrix}.$$

a) Find the eigenvalues λ_1 and λ_2 of A.

b) Find $||A||_2 = \sqrt{\rho(A^T A)}$. Explain why $||A||_2 = \max\{|\lambda_1|, |\lambda_2|\}$ in this case.

4. Assume that you are given $f(a+h)$, $f(a-h)$, $f(a+2h)$, $f(a-2h)$, and $f(a)$.

a) Provide an approximation using these five points (and h) of the 4th derivative $f''''(a)$.

b) Give an estimate of the error of the approximation.

5. **a)** Explain how to solve a linear system $Ax = b$ through the Successive Over-Relaxation method (SOR) as a function of a parameter ω.

b) Show that the SOR may converge **only if** $0 < \omega < 2$.

6. Consider the initial value problem

$$\begin{aligned} y'(x) &= f(x, y) \\ y(0) &= y_0 \end{aligned}$$

One possible numerical scheme to solve this problem is

$$y_{n+1} = y_n + \frac{h}{2}\left[3f(x_n, y_n) - f(x_{n-1}, y_{n-1})\right]$$

where h is the step size.

a) Derive the local truncation error for this scheme.

b) Suppose that the first y_n on the right-hand side of this scheme is replaced by $\frac{1}{2}(y_{n+1} + y_{n-1})$ while all terms involving f remain unaltered. How does this modification affect the local truncation error? What would be the most efficient way to implement the modified scheme?

7. Consider the data points $(-1,-3)$, $(0,1)$, $(1,3)$ and $(2,9)$.

a) Find the Lagrange representation of the cubic polynomial that interpolates between these points. Express the interpolation error in this representation.

b) Find the Newton divided differences representation of the polynomial that interpolates between these points. Express the interpolation error in this representation.