# BASIC EXAM – COMPLEX ANALYSIS

JANUARY 21, 2005

**Provide solutions for Eight of the following Ten problems.** Each problem is worth 10 points. To pass at the Master's level, it is sufficient to have 45 points, with 3 essentially correct solutions; 55 points with 4 essentially correct solutions are sufficient for passing at the Ph.D. level. Indicate clearly which problems you want graded.

NOTATION: We denote by $\mathbb{D}$ the open unit disc, i.e. $\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$.

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1. Use contour integration to verify that for $b > 0$,

$$\int_{-\infty}^{\infty} \frac{\cos x}{x^2 + b^2} dx = \frac{\pi e^{-b}}{b}.$$

Be sure to justify all your steps.

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2. Prove that for any $a > 1$, the equation $z = e^{z-a}$ has exactly one solution in the unit disc $\mathbb{D}$. (Give the full statement of any theorem you use).

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3. Locate the poles of

$$f(z) = \frac{\tan(z)}{z^5},$$

and indicate the order of each pole. Find the principal part, i.e. the coefficients of the negative powers, in the Laurent expansion of $f$ at each pole.

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4. (a) State Morera's theorem.

(b) Use Morera's Theorem to prove that if $f$ is continuous on $\mathbb{C}$ and holomorphic on the set $\Omega = \{z \in \mathbb{C} | \text{Im}(z) \neq 0\}$, then $f$ is holomorphic on $\mathbb{C}$.

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5. (a) State the Schwarz Lemma, then prove it.

(b) Suppose $f$ is a holomorphic mapping of the unit disc $\mathbb{D}$ to itself and that $f$ is not the identity map. Use the Schwarz lemma to prove that $f$ has at most one fixed point in $\mathbb{D}$.

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6. For each part of this problem, indicate whether the statement is true or false. If true, give a proof; if false, provide a counterexample.

(a) There exists a holomorphic function $f$ on the unit disc $\mathbb{D}$ such that $f(1/n) = f(-1/n) = 1/n^3$ for $n = 2, 3, \ldots$.

(b) There exists a holomorphic function $f$ on the punctured unit disc $(\mathbb{D} - \{0\})$ such that $g(z) = e^{f(z)}$ has a simple pole at the origin.

(c) If $f$ is a holomorphic function on the unit disc $\mathbb{D}$ which does not vanish at any point of $\mathbb{D}$, then there exists a function $g$ holomorphic on $\mathbb{D}$ satisfying $g^2 = f$. (i.e. every non-vanishing holomorphic function on $\mathbb{D}$ has a holomorphic square root on $\mathbb{D}$.)

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7. Write down a conformal map that takes the "right-half" of the unit disc $R = \{z \in \mathbb{D} \mid \mathrm{Re}(z) > 0\}$ **onto** the unit disc $\mathbb{D}$.

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8. Use contour integration to prove that

$$\int_0^\infty \frac{x^{1/3}}{1+x^2}\,dx = \frac{\pi}{\sqrt{3}}.$$

Be sure to justify all your steps.

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9. Evaluate

$$\frac{1}{2\pi i}\int_C \frac{\cos^n(z)}{z^3}\,dz$$

where $n \geq 0$ is a non-negative integer, and $C$ is the unit circle $|z| = 1$ traversed counterclockwise once.

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10. (a) Give a careful statement of the Cauchy Inequalities, then prove them by using the Cauchy Integral Formulas.

(b) State Liouville's theorem. Use the Cauchy inequalities to prove Liouville's theorem.

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