University of Massachusetts
Department of Mathematics and Statistics
Advanced Exam in Geometry
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Do 5 out of the following 7 questions. Indicate clearly which questions you want to have graded. *Passing standard:* 70% with three problems essentially complete. **Justify all your answers.**

Problem 1. Let M be a manifold, $f\colon M \to \mathbb{R}$ be a smooth map and $p \in M$ a critical point of f.

a) Define what is meant by the *Hessian* of f at p. If your definition involves choices, show that the end-result is independent of those choices.

b) Let $f\colon S^n \to \mathbb{R}$ be the map

$$f(x) = \sum_{j=1}^{n+1} j\, x_j^2\,; \quad x = (x_1, \ldots, x_{n+1}) \in S^n \subset \mathbb{R}^{n+1}\,.$$

Find the critical points of F.

c) Compute the Hessian, $\operatorname{Hess}(f)$, at each of the critical points obtained above.

Problem 2. Let $H = \{(x, y) \in \mathbb{R}^2 : y > 0\}$ with the Lie group structure defined by the product:

$$(x, y) * (x', y') \;=\; (x + x', yy')\,.$$

a) Find a basis of left-invariant 1-forms on H and define a left-invariant metric on H.

b) Compute the Gaussian curvature of the metric defined in **a)**.

c) Compute the geodesics of H with respect to the metric defined in **a)**.

Problem 3. Let $f\colon \mathbb{RP}^1 \to \mathbb{RP}^n$ be defined by

$$[s : t] \mapsto [s^n : s^{n-1}t : \cdots : st^{n-1} : t^n].$$

a) Show that f defines an embedding of $\mathbb{RP}^1$ in $\mathbb{RP}^n$.

b) Let $\mathcal{T}_n \to \mathbb{RP}^n$ be the tautological line bundle and $\mathbf{1}_n \to \mathbb{RP}^n$ the trivial line bundle over $\mathbb{RP}^n$. What is the relationship between $f^*(\mathcal{T}_n)$ and the bundles $\mathcal{T}_1$ and $\mathbf{1}_1$ over $\mathbb{RP}^1$?

Problem 4. Let (M, g) be an oriented Riemannian manifold and let Ω denote its volume element.

- **a)** Define the Laplace operator $\triangle$ on smooth functions $f \in C^\infty(M, \mathbb{R})$ on M.
- **b)** If M is compact and $f, h : M \to \mathbb{R}$ are smooth functions then prove that

$$\int_M h\,(\triangle f)\,\Omega = -\int_M g(df, dh)\,\Omega$$

- **c)** Show that if M is compact, every function with $\triangle f = 0$ is necessarily constant.

Problem 5. Let G be a compact Lie group and $\sigma \colon G \to G$ a Lie group homomorphism such that $\sigma \circ \sigma = \mathrm{id}$.

- **a)** Prove that $H = \{g : \sigma(g) = g\}$ is a compact Lie subgroup of G.
- **b)** Show that G has a bi-invariant Riemannian metric relative to which σ is an isometry.

Problem 6. Consider the 1-forms in $\mathbb{R}^4$:

$\alpha = (x_1^2 - x_2^2)\,dx_1 - 2x_1x_2\,dx_2 + dx_3\,; \quad \beta = 2x_1x_2\,dx_1 + (x_1^2 - x_2^2)\,dx_2 + dx_4$

For each $p \in \mathbb{R}^4$, let

$$\Delta(p) \ := \ \{v \in T_p(\mathbb{R}^4) : \alpha(p)(v) = \beta(p)(v) = 0\}$$

- **a)** Show that Δ is a 2-dimensional involutive distribution in $\mathbb{R}^4$.
- **b)** Given $p = (0, 0, a, b) \in \mathbb{R}^4$, construct a function $F \colon U \to \mathbb{R}^2$, where $U \subset \mathbb{R}^2$ is an open neighborhood of the origin, such that
  - (a) $F(0, 0) = (a, b)$
  - (b) $\mathrm{Graph}(F) \subset \mathbb{R}^4$ is an integral submanifold of Δ.

Problem 7. We identify S^2 with the Riemann sphere $\mathbb{C} \cup \{\infty\}$ and define $F \colon S^2 \to S^2$ by $F(z) = z^3$ if $z \in \mathbb{C}$, and $F(\infty) = \infty$.

- **a)** Prove that F is a C^∞ map.
- **b)** Show that for all $\omega \in \Lambda^2(S^2)$,

$$\int_{S^2} F^*(\omega) \ = \ 3 \int_{S^2} \omega$$