Department of Mathematics and Statistics
University of Massachusetts

ADVANCED EXAM — DIFFERENTIAL EQUATIONS
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Do five of the following problems. All problems carry equal weight.
Passing level: 75% with at least three substantially complete solutions.

1a) Find an energy function for the system

$$\begin{aligned} x' &= y \\ y' &= x - x^4 \end{aligned} \tag{1}$$

and use this to sketch the phase plane of (1).

1b) Show that for sufficiently small $\varepsilon > 0$ the system of differential equations

$$\begin{aligned} x' &= y \\ y' &= x - x^4 + \varepsilon \sin t \end{aligned}$$

has a periodic solution $(x_\varepsilon(t), y_\varepsilon(t))$ with period 2π that remains in a neighborhood of the origin. (Hint: augment the equations with $\tau' = 1, \tau(0) = 0$ so that $\tau(t) = t$ and rewrite the problem as a fixed point problem for a map.

2) Assume that u is harmonic in Ω.

a) Let φ_ϵ be a standard mollifier; show that

$$u^\epsilon(x) := (\varphi^\epsilon * u)(x) = u(x)$$

for all $x \in \Omega_\epsilon = \{x \in \Omega : dist(x, \partial\Omega) > \epsilon\}$ **(Hint: use the mean value property)**

b) Show that $u \in C^\infty(\Omega)$.

3a) Find a function $K(x,y)$ such that the solutions of the inhomogeneous linear boundary value problem

$$\begin{aligned} u'' &= -u + h(x), \quad 0 < x < L \\ u(0) = 0, & \qquad u'(L) = u(L) \end{aligned}$$

are the integrals $u(x) = \int_0^1 K(x,y)h(y)\,dy$.

3b) Using the representation in **3a)** and the method of succesive approximations, show that if L is sufficiently small and $f(u)$ is a given smooth function of u, the nonlinear boundary value problem

$$\begin{aligned} u'' &= -u + f(u(x)), \quad 0 < x < L \\ u(0) = 0, & \qquad u'(L) = u(L) \end{aligned}$$

has a continuous solution $u(x)$. Explain why u has two continuous derivatives.

4) Suppose $u \in \mathcal{S}(\mathbb{R}^n)$, where $u = u(x), x = (y,z)$ and $y \in \mathbb{R}^k, z \in \mathbb{R}^{n-k}$. Define the (trace) map

$$T : \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^k)$$

as $(Tu)(y) = u(y,0)$.

Show that T can be extended to a bounded linear map $T : H^t(\mathbb{R}^n) \longrightarrow H^s(\mathbb{R}^k)$, provided $s < t - \frac{n-k}{2}$.
Hint: First show that for all $u \in \mathcal{S}(\mathbb{R}^n)$

$$\| Tu \|_{H^s(\mathbb{R}^k)} \leq C(n,k) \| u \|_{H^t(\mathbb{R}^n)}$$

where $C(n,k)$ is a constant depending only on k, n. Also recall that

$$H^s(\mathbb{R}^k) = \{u \in L^2(\mathbb{R}^k) : \ (1 + |\zeta|^2)^{s/2}\hat{u}(\zeta) \in L^2(\mathbb{R}^k)\}$$

5a) Suppose that ℓ is a positive constant, and that $U(x,t)$ is a smooth, classical solution of the PDE on $[0,1]\times[0,T]$ of

$$\begin{aligned} U_{tt} &= U_{xx} - \ell \sin U, \quad 0<x<1 \\ U(0,t) &= U(1,t) = 0, \quad 0 \le t \le T \\ (U(x,0), U_x(0,t)) &= (f(x), g(x)) \end{aligned} \tag{2}$$

Consider the system of ODE's

$$\begin{aligned} u_j' &= v_j \\ v_j' &= -\ell \sin u_j + N^2(u_{j+1} - 2u_j + u_{j-1}) + E_j(t), \end{aligned} \tag{3}$$

for $1 \le j \le N$, where $u_j \equiv v_j \equiv 0$ for $j=0$ and $j=N+1$, and where $E_j(t)$ are specified functions. Let $x_j = j/N$ for $1 \le j \le N$ and define $u_j(t) = U(x_j,t)$ and $v_j(t) = U_x(x_j,t)$. Show that $u_j(t), v_j(t)$ satisfy a system of the form (3) on $0 \le t \le T$ for some functions $E_j(t)$ that satisfy the estimate: $|E_j(t)| \le K/N^2$ for some constant $K>0$ depending only on U and its derivatives up to fourth order.

5b) Now suppose that $\bar{u}_j(t), \bar{v}_j(t)$ is the solution of the homogeneous system

$$\begin{aligned} u_j' &= v_j \\ v_j' &= -\ell \sin u_j + N^2(u_{j+1} - 2u_j + u_{j-1}), \end{aligned} \tag{4}$$

obtained as a formal approximation to (2) for large N. Calculate the eigenvalues of the linearization of this system at the rest point $u_j = v_j = 0$ for all j. What, if anything, can be concluded from this calculation ?

5c) Find an energy function $E(u,v)$ for the homogeneous system in (4), and use this to show that the rest point in at the origin is stable for all N. (Hint: what does the energy for the PDE look like?)

6 Suppose u is a smooth solution of

$$\begin{cases} u_t - \Delta u = f & in \quad \mathbb{R}^n \times (0,T) \\ u(x,0) = g(x) & in \quad \mathbb{R}^n \end{cases},$$

Show that if u and all it's spatial derivatives decay at infinity, then

$$\max_{0 \le t \le T} \int_{\mathbb{R}^n} |\bigtriangledown u|^2 dx + \int_0^T \int_{\mathbb{R}^n} (u_t^2 + |\bigtriangledown^2 u|^2) dx dt \le C \left[\int_0^T \int_{\mathbb{R}^n} f(x,t)^2 \, dx dt + \int_{\mathbb{R}_n} |\bigtriangledown g|^2 dx \right]$$