University of Massachusetts
Department of Mathematics and Statistics
Advanced Exams in Geometry
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Do 5 out of the following 7 questions. Indicate clearly what questions you want to have graded. Passing standard: 70% with three problems essentially complete. Justify all your answers.

Problem 1. Show that over the circle S^1 there are exactly two isomorphism classes of rank r real vector bundles.

Problem 2. Consider the Helicoid

$$f : \mathbb{R}^2 \to \mathbb{R}^3 : (u, v) \mapsto (u \cos v, u \sin v, v)\,.$$

(1) Show that f is an embedding of $\mathbb{R}^2$ into Euclidean 3-space.
(2) Calculate the induced Riemannian metric of this embedding and the distance from $(0, 0, 0)$ to $(100, 0, 0)$, and from $(0, 0, 0)$ to $(0, 0, 2\pi)$ on the Helicoid in the induced metric.
(3) Compute the Gauss and mean curvature functions of the Helicoid.

Problem 3. Prove that any Lie group is an orientable manifold.

Problem 4. A manifold is called k-parallelizable if it admits vector fields $X_1, ..., X_k$ which are linearly independent at each point. Show that any odd-dimensional sphere S^n is 1-parallelizable, and that if $n = 4m - 1$, it is actually 3-parallelizable. (Hint: Think of the real division algebras $\mathbb{C}$ and $\mathbb{H}$.)

Problem 5. Consider the doubly-periodic 1-form

$$\alpha = \cos^{12} x\; dx + \sin^{12} y\; dy$$

on $\mathbb{R}^2$.

(1) Show there is a unique $f : \mathbb{R}^2 \to \mathbb{R}$ such that $df = \alpha$ and $f(0, 0) = 0$.
(2) Regarding α as a 1-form on the torus $T^2 = \mathbb{R}^2/2\pi\mathbb{Z}^2$, determine whether or not there is a $g : T^2 \to \mathbb{R}$ such that $dg = \alpha$. (Hint: Compute the deRham class $[\alpha] \in H^1(T^2, \mathbb{R})$).

Problem 6. Let M be a smooth compact and oriented n-dimensional manifold with boundary ∂M.

(1) Let f be a smooth function on M, ω a smooth $(n-1)$-form on M, and suppose $f\omega = 0$ on ∂M. Prove that
$$\int_M df \wedge \omega = -\int_M f d\omega\,.$$
(2) Suppose f is a harmonic function which vanishes on the boundary, i.e., the Laplacian $\Delta f = *d * df = 0$ on M and $f = 0$ on ∂M. Prove $f = 0$ on M.

Problem 7. For a complex $n \times n$-matrix we define $A^* := \bar{A}^T$ to be the conjugate transposed matrix. Consider the set

$$\mathbf{SU}(n) := \{A \in \mathbf{GL}(n, \mathbb{C})\,;\; A^* = A^{-1}\,,\, \det A = 1\}\,.$$

Show that $\mathbf{SU}(n)$ is a real Lie group, determine its Lie algebra and calculate its dimension.