DEPARTMENT OF MATHEMATICS AND STATISTICS
UNIVERSITY OF MASSACHUSETTS
ADVANCED EXAM - MATHEMATICAL STATISTICS
TUESDAY, JANUARY 22, 2002

Do all six problems. All problems are equally weighted. Seventy points are needed to pass at the Ph.D. level.

(1) Let X and Y be independent, non-negative, real-valued random variables. Their respective cumulative distribution functions are $F(x)$ and $G(x)$ and their respective probability densities (with respect to Lebesgue measure) are $f(x) > 0$ and $g(x) > 0$. Define $Z = \min(X, Y)$ and

$$I = 0, \text{ if } X \geq Y; I = 1, \text{ if } X < Y.$$

(a) Derive the joint distribution of (Z, I).
(b) Verify the following:

$$\frac{-(d/dx)P(Z \geq x, I = 1)}{P(Z \geq x)} = \frac{f(x)}{1 - F(x)}.$$

(2) Let $X_n, n = 1, 2, \cdots,$ and X be real-valued random variables defined on the same probability space.
(a) Give an example in which $X_n \Rightarrow X$ in distribution, but $X_n \not\to X$ in probability.
(b) Suppose $X_n \Rightarrow X$ in distribution and $X = c$, where c is a deterministic number. Show that $X_n \to c$ in probability.
(c) Give an example in which $X_n \Rightarrow X$ in distribution, but

$$\lim_{n\to\infty} E[X_n] \neq E[X].$$

(d) Let $X_1, X_2, \ldots$ be i.i.d. with mean 1 and finite variance σ^2. Show that $(1/n)\sum_{i=1}^{n} X_i X_{i+1}$ converges in probability, and identify the limit.

(3) Let X and Y be bounded, real-valued random variables on a probability space $(\Omega, \mathcal{F}, P)$. Assume $\mathcal{G}$ is a sub-sigma field of $\mathcal{F}$. Prove that

$$E[YE[X|\mathcal{G}]] = E[XE[Y|\mathcal{G}]]$$

using the definition of conditional expectation.

(4) Let $X_1, \cdots, X_n$ be a random sample from an exponential distribution with unknown mean θ, i.e., $p(x) = \theta^{-1}\exp(-x/\theta)$, for $x > 0$ and $\theta > 0$.
(a) Find the MLE of the 75th percentile, say q, of the distribution.
(b) Determine whether or not this estimator is unbiased.

(c) Calculate the mean squared error (MSE) of the estimator in terms of n and θ. Does $MSE \to 0$ as $n \to \infty$?

(5) Let $X_1, \ldots, X_m$ and $Y_1, \ldots, Y_n$ be two independent random samples from $N(\mu_1, \sigma_1^2)$ and $N(\mu_2, \sigma_2^2)$, respectively, where all parameters are unknown. Let $\overline{X}_1$ and $\overline{X}_2$ denote the sample means; S_1^2 and S_2^2 denote the sample variances of the two samples, respectively.

(a) Write down (without proof) the MLEs, $\hat{\mu}_1, \hat{\sigma}_1^2, \hat{\mu}_2, \hat{\sigma}_2^2$ for $\mu_1, \sigma_1^2, \mu_2, \sigma_2^2$, respectively.

(b) Derive the α-level likelihood ratio test for $H_0 : \sigma_1^2 = \sigma_2^2$ *against* $H_1 : \sigma_1^2 \neq \sigma_2^2$. The resulting test should be expressed in terms of a well-known statistic. (You need not derive the usual MLEs, but you do need to justify the likelihood ratio test.)

(6) Let $X_1, \ldots, X_n$ be i.i.d. random variables from Binomial(r, θ), where $0 < \theta < 1$ and $r \geq 1$ is an integer.

(a) Justify that $T = \sum_{i=1}^{n} X_i$ is a complete and sufficient statistic for θ.

(b) Write $q = \Pr[X_1 \leq 1]$ in terms of θ, and define a random indicator U which is an unbiased estimator for q.

(c) Use the properties of T and the Rao-Blackwell Theorem to find the UMVU estimator of q.