This exam consists of eight equally weighted problems (ten points each): a passing grade is 65% (52/80), including at least five “essentially correct” problems ($\geq 7.5/10$).

Clearly show your work, explicitly stating or naming results that you use; justify the use of named theorems by verifying necessary conditions.

Please work legibly and clearly label each page/file of your exam with your name. You may use the back of the page for additional work.
1. Calculate the following limits and justify your calculations:

(a) \( \lim_{n \to \infty} \int_0^\infty (1 + (x/n))^{-n} \sin(x/n) \, dx \).

(b) \( \lim_{n \to \infty} \int_0^1 (1 + nx^2)(1 + x^2)^{-n} \, dx \).

(c) \( \lim_{n \to \infty} \int_0^\infty n \sin(x/n)[x(1 + x^2)]^{-1} \, dx \).
2. Let \((X, \mathcal{M}, \mu)\) be a finite measure space with \(\mu(X) > 0\) and let \(\varphi\) be a nonnegative, Borel-measurable function mapping \(X\) into \([0, \infty)\). Assume that \(\varphi \not\equiv 0\) a.e. For \(A \in \mathcal{M}\), define

\[
\lambda(A) = \int_A \varphi \, d\mu.
\]

(a) Prove that there exists \(A \in \mathcal{M}\) such that \(\lambda(A) > 0\). This proves that \(\lambda\) is nontrivial.

(b) Prove that \(\lambda\) is a measure on \(\mathcal{M}\) and that \(\lambda \ll \mu\).

(c) Let \(h\) be any nonnegative, Borel-measurable function mapping \(X\) into \([0, \infty)\). Prove that

\[
\int_X h \, d\lambda = \int_X \varphi \, h \, d\mu.
\]
3. If $a, b > 0$, let

$$f(x) = \begin{cases} 
  x^a \sin(x^b) & \text{for } 0 < x \leq 1, \\
  0 & \text{if } x = 0.
\end{cases}$$

Prove that $f$ is of bounded variation in $[0, 1]$ if and only if $a > b$. 
4. Let \((X, \mathcal{M}, \mu)\) and \((Y, \mathcal{N}, \nu)\) be two measure spaces and \(K, K_1, K_2 : X \times Y \to \mathbb{R}\) such that \(|K| \leq K_1 K_2\) and

\[
\|K_1(x, \cdot)\|_{L^q(Y)} \leq C_1 \mu \text{ almost everywhere and}
\]

\[
\|K_2(\cdot, y)\|_{L^p(X)} \leq C_2 \nu \text{ almost everywhere, with } \frac{1}{p} + \frac{1}{q} = 1, \ p, q, \in (1, \infty).\]

Prove that the linear operator

\[
(Tf)(x) = \int_Y K(x, y) f(y) d\nu(y)
\]

is bounded from \(L^p(Y)\) to \(L^p(X)\), with \(\|T\| \leq C_1 C_2\).
5. Let $H$ be an infinite dimensional Hilbert space with norm $\| \cdot \|$ and inner product $\langle \cdot , \cdot \rangle$.

(a) Prove that the unit sphere in $H$ (that is $\{ x \in H \mid \| x \| = 1 \}$) is not compact.

(b) Show that if $\{ x_n \}$ is a sequence of unit vectors in $H$ then there is a subsequence $\{ x_{n_j} \}$ and an element $x \in H$ such that for all $y \in H$

$$\lim_{j \to \infty} \langle x_{n_j}, y \rangle = \langle x, y \rangle.$$ 

Hint: Let $y$ run through a basis for $H$ and use a diagonalization argument. One can then defined $x$ by giving its series expansion with respect to the chosen basis.
6. In this problem \( \hat{f} \) denotes the Fourier transform of \( f : \mathbb{R}^d \to \mathbb{R} \).

(a) Prove that if \( f \in L^1 \) then \( \hat{f} \) is continuous and \( \lim_{|\xi| \to \infty} \hat{f}(\xi) = 0 \).

(b) Let \( d = 1 \) (that is, \( f : \mathbb{R} \to \mathbb{R} \)). Prove that there is a constant \( C > 0 \) such that for any Schwartz function \( f \)

\[
\left( \int_{\mathbb{R}} x^2 |f(x)|^2 \, dx \right) \left( \int_{\mathbb{R}} \xi^2 |\hat{f}(\xi)|^2 \, d\xi \right) \geq C \|f\|^2_{L^2}.
\]

Hint: Note that \( |f(x)|^2 = |f(x)|^2 \frac{d}{dx} x \), and use integration by parts.
7. Let $H$ be a Hilbert space with inner product $\langle \cdot, \cdot \rangle$, and recall that a bounded linear operator $T : H \to H$ is called compact if for every bounded sequence $\{x_n\}$ in $H$, $\{Tx_n\}$ has a convergent subsequence.

(a) Give an example of a compact operator and an example of a non-compact operator, both defined on an infinite dimensional Hilbert space.

(b) Prove that if $T$ is compact then so are $T^*$ and $T^*T$.

(c) Prove that if $T$ is compact then there exists orthonormal sets (neither of which is necessarily a basis) $\{\phi_n\}_{n=1}^N$ and $\{\psi_n\}_{n=1}^N$ ($N$ may be a positive integer or $\infty$) and positive real numbers $\{\lambda_n\}_{n=1}^N$ with $\lambda_n \to 0$ such that

$$T = \sum_{n=1}^N \lambda_n \langle \psi_n, \cdot \rangle \phi_n.$$  

Hint: Use the previous part (even if you couldn’t prove it) and the spectral theorem for compact self-adjoint operators.
8. Let \((X, \mathcal{A}, \mu)\) be a measure space. Let \(f\) and \(f_n, \ n \geq 1\) be measurable functions on \(X\). Recall that \(f_n\) is said to converge to \(f\) in measure if for every \(\epsilon > 0\),

\[
\mu(\{x \mid |f_n(x) - f(x)| \geq \epsilon\}) \to 0 \quad \text{as} \ n \to \infty.
\]

(a) Suppose that \(\mu(X) < \infty\). Prove that \(f_n \to f\) a.e implies \(f_n \to f\) in measure.

(b) Prove that the converse of (b) is false even for \(\mu(X) < \infty\). (Hint. Let \(X = [0, 1]\) and consider the (double) sequence \(f_{m,k}(x) = 1_{E_{m,k}}(x)\) \((m, k \in \mathbb{N})\), where \(E_{m,k} := [\frac{m-1}{k}, \frac{m}{k}]\)).