Passing Standard: The passing standard is 70% with essentially correct solutions to five problems. Show all your work and justify your answers carefully. All rings are commutative and contain the identity, and all ring homomorphisms preserve the identity.

Part I. Group Theory

1. For any Abelian group $G$, denote by $G^*$ the set of all group homomorphisms $G \rightarrow \mathbb{C}^\times$. It is a fact that $G^*$ is an Abelian group (you do not have to verify this).

Show that if $A$ is a finite Abelian group, then $A \cong A^*$.

Please work out each problem on a separate sheet of paper. Single-sided only.

2. Let $p$ be a prime number, and let $P_1, P_2$ be Sylow $p$-subgroups of a finite group $G$.

(a) If $P_1 \subset N_G(P_2)$, show that $P_1 = P_2$.

(b) If $N_G(P_1) = N_G(P_2)$, show that $P_1 = P_2$.

Note: For any group $G$ and any subgroup $H$ of $G$, denote by $N_G(H) := \{g \in G : gHg^{-1} = H\}$ the normalizer of $H$ in $G$.

Part II. Commutative Algebra

3. Let $A$ be a commutative ring. Let $M$ be an $A$-module, and let $N$ be an $A$-submodule of $M$. For each of the following statements, give a proof if it is true, otherwise give a counter-example.

   (a) If $N$ and $M/N$ are free $A$-modules, then $M$ is also a free $A$-module.

   (b) If $M$ is a finitely generated $A$-module, then at least one of $N$ or $M/N$ is also a finitely generated $A$-module.

Please work out each problem on a separate sheet of paper. Single-sided only.

4. Let $A$ be an integral domain.

   (a) Let $S$ be a multiplicatively closed subset of $A$ with $0 \not\in S$. If $A$ is an UFD (unique factorization domain), show that $S^{-1}A$ is also an UFD.

   (b) Let $\alpha \in A$ be such that the principal ideal $\alpha A$ is a non-zero prime ideal, and let $S = \{\alpha^n : n \in \mathbb{N}\}$. If $S^{-1}A$ is an UFD, show that $A$ is also an UFD.

Please work out each problem on a separate sheet of paper. Single-sided only.

5(a). Determine all conjugacy classes of matrices over $\mathbb{Q}$ with characteristic polynomial $(x + 1)^3(x^2 + 1)$. Show your work!

(b) Repeat the same analysis, with $\mathbb{Q}$ replaced by the finite field $\mathbb{F}_2$.

Note/Hint: (1) We have provided all the information needed to solve this problem. (2) While this problem has two parts, there is a significant overlap between the two parts.
**PART III. FIELD THEORY AND GALOIS THEORY**

6. Determine the Galois group of $x^5 - 5$ over $\mathbb{Q}$. *Show your work!*

7. Let $K/L$ be a (finite) Galois extension, and let $L \subset F \subset K$ be an intermediate subfield. Define

$$H := \{ g \in \text{Gal}(K/L) : g(F) = F \}.$$ 

Show that $H$ is the normalizer of $\text{Gal}(K/F)$ in $\text{Gal}(K/L)$.

*Note: See problem #2 for the definition of the normalizer of a subgroup.*

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