

DEPARTMENT OF MATHEMATICS AND STATISTICS
UNIVERSITY OF MASSACHUSETTS AMHERST
APPLIED MATHEMATICS EXAM
AUGUST 2021

Do all six problems. All problems carry equal weight.

Passing level: 65% with at least three substantially correct.

1. An object with a small mass m is attached to a spring. It was at the resting position initially, and then hit by a force I (in the direction that the spring would normally stretch). Assuming that the resistance is proportional to the object's speed with constant $\alpha > 0$, and the nonlinear damping force (of the spring) is given by κy^3 , where y is the displacement of the object from its resting position. Write down the equation for the dynamics, and find a leading order perturbation approximation.
2. An example of rabbit-sheep model is given by ($x > 0$ and $y > 0$):

$$\dot{x} = x(3 - x - y)$$

$$\dot{y} = y(5 - 2x - y)$$

- (a) Find all the equilibria, classify their stability and sketch the phase plane.
 - (b) Find the limit cycles if there is any in this system, or explain why there can not be any.
3. Suppose that a spherical object is shot through the air on Mars at high speed (which may be supersonic). Using only dimensional analysis to determine how the drag force, F_D , on the object depends on the 'relevant' parameters: object radius R , object velocity V , Mars air density ρ , and the sound speed c . (There will be two dimensionless quantities in this formula.)
 4. Consider the system

$$\dot{x} = \lambda x + x^3 - x^5$$

with real parameter λ . Find all equilibria as functions of λ , sketch the bifurcation diagram, indicate and classify the bifurcation point.

5. Compute the monodromy matrix and the Floquet multipliers for the system $x'(t) = A(t)x(t)$, where $A(t)$ is defined by

$$A(t) = \begin{bmatrix} -2 & \sin(t) \\ 0 & -1 \end{bmatrix}.$$

6. Find the leading order uniform approximation to to boundary value problem

$$\epsilon y'' + y' + y^2 = 0 \quad y(0) = 2, y(1) = \frac{1}{2},$$

where $\epsilon > 0$ is a small constant.