UMass Amherst Algebra Advanced Exam

Monday 8/17/20, 10:00AM–1:00PM.

Instructions: Show all your work and justify your answers carefully. The passing standard is 70% with essentially correct solutions to 5 problems.

1. Group theory

Q1. Let G be a group of order 12.

(a) Prove that G has a normal Sylow subgroup.

(b) Suppose in addition that G is nonabelian and contains an element of order 4. Describe G in terms of generators and relations.

Q2. Let G be a finite group acting on a finite set X. For any $g \in G$, let $X^g = \{x \in X \mid g \cdot x = x\}$ be the fixed point set of g. Prove that the number of G-orbits in X is $|G|^{-1} \sum_{g \in G} |X^g|$.

[Hint: First prove $\sum_{x \in X} |G_x| = \sum_{g \in G} |X^g|$, where G_x denotes the stabilizer of $x \in X$ under the action of G.]

2. Commutative Algebra

Q3.

(a) Let $R = \mathbb{Z}[\sqrt{-2}] = \{a + b\sqrt{-2} \mid a, b \in \mathbb{Z}\} \subset \mathbb{C}$. Prove that R is a unique factorization domain.

(b) Let $S = \mathbb{Z}[\sqrt{-13}] = \{a + b\sqrt{-13} \mid a, b \in \mathbb{Z}\} \subset \mathbb{C}$. Prove that S is *not* a unique factorization domain.

Q4. Enumerate the conjugacy classes of elements of order dividing 6 in (a) $\mathrm{GL}_3(\mathbb{C})$ and (b) $\mathrm{GL}_3(\mathbb{Q})$.

Q5. Let R be a commutative ring with 1 and J an R-module. We say J is *injective* if the following property is true: Suppose $g\colon A \to B$ is an injective homomorphism of R-modules. Then for any homomorphism $f\colon A \to J$, there exists a homomorphism $h\colon B \to J$ such that $f = h \circ g$.

(a) Show that a $\mathbb{Z}$-module D is injective if and only if for every $d \in D$ and positive integer n, the equation $nx = d$ has a solution in D.

[Hint: To prove sufficiency of the condition, by Zorn's lemma we may assume that $B/g(A)$ is generated by a single element (with notation as in the definition of injective above).]

(b) A $\mathbb{Z}$-module satisfying the condition of part (a) is called *divisible*. Show that if D is divisible and $\theta\colon D \to E$ is a homomorphism of $\mathbb{Z}$-modules, then the image $\theta(D)$ of θ is also divisible.

(c) Show that for any finitely generated $\mathbb{Z}$-module A there exists a divisible $\mathbb{Z}$-module D and an injective homomorphism $A \to D$.

3. Field theory and Galois theory

Q6.

(a) Compute the minimal polynomial of $\alpha = \sqrt{2} + \sqrt{-3}$ over $\mathbb{Q}$.

(b) Prove that the extension $\mathbb{Q} \subset \mathbb{Q}(\alpha)$ is Galois and determine all the intermediate fields.

Q7. Describe a Galois extension of $\mathbb{Q}$ such that the Galois group is cyclic of order 5.

[Hint: Let p be a prime and $\zeta = e^{2\pi i/p}$. What is the Galois group of the cyclotomic field $\mathbb{Q}(\zeta)$ over $\mathbb{Q}$?]