Advanced Calculus/Linear algebra basic exam

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Instructions: Do 7 of the 8 problems. Show your work. The passing standards are:

- Master's level: 60% with three questions essentially, complete (including one question from each part);
- Ph.D. level: 75% with two questions from each part essentially complete.

Advanced Calculus

1. Compute the following integrals and limits.
   (a) Find the derivative of the function $f(x) = \tan\left(\sqrt{\frac{1-x}{1+x}}\right)$.
   (b) Evaluate $\displaystyle\int x^2 \sin(x^3)dx$.
   (c) Switch the order of integration of $\displaystyle\int_0^{2\pi}\int_0^1\int_{-1}^{-2r+1} (r^2\cos^2\theta + z^2)r^2 \; dzdrd\theta$ to $drd\theta dz$. **Do not evaluate.**
   (d) The triple integral in (c) calculates a moment of inertia of a solid.
      1. What is that solid?
      2. Around what axis does the solid revolve?
      3. What is the density of the solid?

2. Find the radius of converge of each of the following power series:
   (a) $\displaystyle\sum_{n=0}^{\infty} \frac{n^3}{3^n} z^n$,
   (b) $\displaystyle\sum_{n=0}^{\infty} \frac{2^n}{n!} z^n$.

3. Among all the planes that are tangent to the surface $xy^2z^2 = 1$, find the one(s) that are farthest away from the origin.

4. Compute the flux of $\mathbf{F} = \langle x+y, y, z\rangle$ across the part of the cone $y = \sqrt{4x^2+4z^2}$ that lies between the planes $y = 0$ and $y = 4$, oriented in the direction of the negative y-axis, in two ways:
   (a) directly,
   (b) by using the divergence theorem.

Turn the page

Linear Algebra

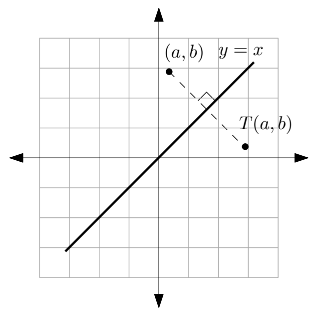
5. Consider the matrix $A = \begin{pmatrix} 1 & 2 & -1 \\ 2 & 2 & c \\ 3 & 5 & -2 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix}$.

   (a) Characterize the values c such that the matrix has a nontrivial nullspace (kernel).

   (b) For $c = 1$, find all solutions $\mathbf{x}$ to the system $A\mathbf{x} = \mathbf{b}$.

   (c) For $c = 0$, find all solutions $\mathbf{x}$ to the system $A\mathbf{x} = \mathbf{b}$.

6. Consider the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ that reflects about the line $y = x$.



   (a) What is the image $T(1,0)$ and $T(0,1)$.

   (b) Find the 2×2 matrix A representation of T with respect to the basis $\{(1,0),(0,1)\}$.

   (c) Find the eigenvectors and the corresponding eigenvalues of T.

   (d) Diagonalize A.

   (e) Give an example of a 2×2 nonzero matrix B that has no eigenvectors and justify your answer.

7. (a) Find an orthogonal basis for the span of the following three vectors $S_2 = \{(1,0,1),(0,1,1),(1,3,3)\}$ and write $(1,1,2)$ in terms of such a basis.

   (b) Let $W = \operatorname{span}((1,1,1))$. Find a basis of the orthogonal complement $W^\perp$ in $\mathbb{C}^3$.

8. Let V be a vector space and $T : V \to V$ be a linear transformation.

   (a) Show that $\{x \mid T(x) = x\}$ is a subspace of V.

   (b) Show that if $T^2 = T$, then for any $x \in V$ we have that $x - T(x)$ is in the nullspace (kernel) of T denoted by $\ker(T)$.

   (c) Show that if $T^2 = T$ then $V = \{x \mid T(x) = x\} \oplus \ker(T)$.

   (d) Characterize all linear transformations T such that $T^2 = T$.