BASIC EXAM: TOPOLOGY, SUMMER 2017

Answer five of the seven questions. Indicate clearly which five questions you want graded. Justify your answers. Passing standard: For Masters level, 60% with two questions essentially complete. For Ph.D. level, 75% with three questions essentially complete.

Problem 1 Consider the following topologies on the real line $\mathbb{R}$:
(i) trivial topology, (ii) discrete topology, (iii) finite complement topology. For each topology, determine, with explanations, which one of the following functions from $\mathbb{R} \to \mathbb{R}$ (both the domain and the range taken with the same topology)

$$f(x) = x^4 \ , \quad g(x) = e^x \ , \quad h(x) = cos(x)$$

are **(a)** continuous, **(b)** open maps, **(c)** embeddings.

Problem 2 Prove that a metric space has a countable dense subset if and only if it has a countable basis for its topology.

Problem 3 Prove that none of the following spaces are *homeomorphic* to each other: $S^1 \times \mathbb{R}$ $S^1 \times [0,1]$, $S^1 \times S^1$, S^2 (Here S^n denotes the n-dimensional unit sphere in $\mathbb{R}^{n+1}$.)

Problem 4 Let X be the union of the unit 2–sphere S^2 and the arc

$$A = \{(0,0,t) \mid -1 \leq t \leq 1\}$$

in $\mathbb{R}^3$, equipped with the subpsace topology. Show that $\pi_1(X) \cong \mathbb{Z}$.

Problem 5 Let (X, d) be a metric space, and let $f : X \to X$ be a continuous function without any fixed points.

(1) If X is compact, show that there exists an $\epsilon > 0$ so that $d(x, f(x)) > \epsilon$ for all $x \in X$.
(2) Give an example to show that this is not true if X is not compact.

Problem 6 Let $p : \tilde{X} \to X$ be a covering map. Let Y be connected and $y_0 \in Y$. Let $f, g : Y \to \tilde{X}$ be continous maps such that:

- $f(y_0) = g(y_0)$, and
- $p \circ f = p \circ g$. Prove that $f = g$.

Problem 7

(1) Define the one-point compactification of a space.
(2) Let X be a connected locally compact space. Prove that X is not homeomorphic to its one-point compactification.