UNIVERSITY OF MASSACHUSETTS
Department of Mathematics and Statistics
Basic Exam - Statistics
Wednesday, August 30, 2017

Work all problems. 60 points are needed to pass at the Masters Level and 75 to pass at the Ph.D. level. Each answer is worth approximately the same number of points.

1. Let $(x_1, Y_1), \ldots, (x_n, Y_n)$ be n pairs of independent samples. Consider the Poisson regression model with the probability mass function, denoted as $Y_i \sim$ Poisson$(x_i\beta)$,

$$p(Y \mid x, \beta) = \exp(-x\beta)\frac{(x\beta)^Y}{Y!}$$

where $i = 1, \ldots, n$ and $x_1, \ldots, x_n$ are positive known constants.

   (a) Find the maximum likelihood estimator (MLE) for β, denoted as $\hat{\beta}$.

   (b) Compute the mean and variance of $\hat{\beta}$.

   (c) Assume that β has the gamma prior density

$$p(\beta \mid a, b) = \frac{a^{ab}}{\Gamma(ab)}\beta^{ab-1}\exp(-a\beta),$$

   where $a > 0$, $b > 0$ and $E(\beta) = b$.
   Find the posterior density of β given $\mathbf{Y}$.

   (d) Compute the posterior mean of β. When $a \to 0$, explain the behavior of the posterior mean. [Hint]: show that the posterior mean of β is the weighted average of the prior mean and the MLE, $\hat{\beta}$.

2. Let $\mathbf{X} = (X_1, \ldots, X_n)$ be a random i.i.d. sample from the following probability density function

$$g(x \mid \theta) = \frac{1}{\theta}x^{(1-\theta)/\theta},$$

where $\theta > 0$ and $0 \leq x \leq 1$.

   (a) Show that $S(\mathbf{X}) = -2\sum_{i=1}^{n} \log X_i$ is a minimal sufficient statistic for θ.

   (b) Derive the exact distributions of the following quantities, $S_i = -2\log X_i$, $S(\mathbf{X})$ and $S(\mathbf{X})/\theta$.

   (c) Find a two-sided 95% confidence interval for θ based on $S(\mathbf{X})/\theta$.

   (d) Discuss what happens to the expected length of the confidence interval for θ obtained in part (c) as $n \to \infty$.

3. Let $X_1, \ldots X_n$ be a random sample from a normal distribution with mean 0 and variance σ^2 unknown.

   (a) Find the MLE of σ^2. Is this an UMVUE? Why or why not? If not, find the UMVUE.

   (b) Find an exact $(1-\alpha)$-level confidence interval for σ^2.

   (c) Find an exact $(1-\alpha)$-level confidence interval for σ.

   Now let $Z = X_1$ and $W = X_1^2$.

   (d) Find $E(Z)$ and $E(W)$.

   (e) Find $E(ZW)$.

   (f) Find $Cov(Z, W)$.

   (g) Are Z and W independent? Why or why not?

4. Suppose $X, Y \sim^{iid} Uniform(0, \theta)$, with θ unknown. We wish to test the hypotheses H_1: $\theta > 1$ against the null H_0: $\theta \leq 1$.

   Let $Z = X + Y$. One family of tests rejects if $Z > C$ for some C. Consider C=1.9

   (a) What is the type-I error rate of this test?

   (b) Consider observing $X = 1.5, Y = .2$. Would this test reject or not reject?

   (c) Considering your answer to the previous part, do you consider this test to be a sensible test? If so, why? If not, propose an alternative family of tests that you would prefer.

   (d) For the test you prefer (either the test of form $Z > C$ or the test you proposed in the previous part), find the exact numerical critical value for a size $\alpha = .05$ test of H_1 vs. H_0.