DEPARTMENT OF MATHEMATICS AND STATISTICS
UNIVERSITY OF MASSACHUSETTS
BASIC EXAM - PROBABILITY
Fall 2017

Work all problems. Show all work. Explain your answers. 60 points are needed to pass at the Masters level and 75 at the Ph.D. level. Each question is worth 20 points.

1. Let X_1 and X_2 have the joint probability density function given by

$$f(x_1, x_2) = \begin{cases} c(2 - x_2), & 0 \le x_1 \le x_2 \le 2 \\ 0, & \text{otherwise} \end{cases}$$

For the following parts to this question, you can leave your solutions in terms of integrals with explicit limits. You do not need to give the final numerical results.

(a) Find c.

(b) Draw a picture to show the x_1 and x_2 values where the density is non-zero.

(c) Find the marginal density functions for X_1 and X_2.

(d) Are X_1 and X_2 independent? Explain why or why not.

(e) Find the conditional density function of X_2 given $X_1 = x_1$.

(f) Find $P(X_2 \ge 1.7 \mid X_1 = 0.6)$

2. Consider a random variable Y with the following density:

$$f_Y(y) = \begin{cases} ce^{-y^2/2}, & -\infty < y < \infty \\ 0, & \text{otherwise} \end{cases}$$

(a) Find c and explain how you found it.

(b) Find the mean of Y and explain how you found it.

(c) Find the variance of Y and explain how you found it.

3. Three methods, A, B and C, are available for teaching a certain industrial skill. A company currently has a limited number of trained instructors for each method. 50% of the time, Method A is used; 30% of the time, Method B is used; and for the remainder of the time, Method C is used. Given that Method A is used, the failure rate is 10%; given that Method B is used, the failure rate is 6%; and given that Method C is used, the failure rate is 8%.

(a) For a randomly selected worker, what is the probability that the worker failed to learn the skill?

(b) For a randomly selected worker, what is the probability that the worker was taught using Method B and failed to learn the skill?

(c) For a randomly selected worker, if the worker failed to learn the skill, what is the probability that the worker was taught by Method B?

(d) n workers are randomly selected. Let X_1 be the number of workers among the n selected that were taught by Method A and failed to learn the skill, X_2 be the number of workers among the n selected that were taught by Method A and learned the skill, X_3 be the number of workers among the n selected that were taught by Method B and failed to learn the skill, X_4 be the number of workers among the n selected that were taught by Method B and learned the skill, X_5 be the number of workers among the n selected that were taught by Method C and failed to learn the skill, and X_6 be the rest. What is the joint probability distribution of X_1, X_2, X_3, X_4, X_5, and X_6? Give the name and the parameters of the distribution.

(e) Based on the previous part, what is the conditional distribution of X_3 given $X_1+X_3+X_5 = r$ for some $0 < r < n$?

4. Suppose we have a collection of Poisson random variables indexed by time, t, such that for any given $t > 0$ we have $N_t \sim \text{Poisson}(rt)$ with probability mass function

$$\Pr(N_t = n) = \frac{(\exp(-rt))(rt)^n}{n!}, \quad n = 0,1,2,...$$

(a) For a fixed t, what is $E\left(N_t / t\right)$? Show your work.

(b) For a fixed t, what is $\text{Var}\left(N_t / t\right)$? Show your work.

(c) What can you say happens to the distribution of N_t / t as t goes to infinity? Note: look at the expected value and variance and relate those to the location of the center of mass and the spread around the center of mass.

5. Consider $X_1, X_2, ..., X_n \sim$ i.i.d. Bernoulli(p).

(a) Let $\bar{X}_n = \sum_{i=1}^{n} X_i / n.$ Find a value of τ in terms of p such that $n^{1/2}\left(\bar{X}_n - p\right)/\tau$ converges in distribution to a standard normal. Justify your answer.

(b) Show that $\hat{\tau}_n = \sqrt{\bar{X}_n\left(1-\bar{X}_n\right)}$ converges in probability to τ.

(c) Show that $n^{1/2}\left(\bar{X}_n - p\right)/\hat{\tau}_n$ converges in distribution to a standard normal.