

ADVANCED CALCULUS/LINEAR ALGEBRA BASIC EXAM

DEPARTMENT OF MATHEMATICS AND STATISTICS
UNIVERSITY OF MASSACHUSETTS AMHERST
SEPTEMBER 7, 2017

Do all 7 problems. **Show your work.** The passing standards are:

- Master's level: 60% with three questions essentially complete (including one from each part);
- Ph.D. level: 75% with two questions from each part essentially complete.

Linear Algebra

- (1) (a) Let $T: V \rightarrow V$ be a linear transformation on a vector space V over a field F and suppose that $V = \text{im } T + \text{ker } T$; that is, V is spanned by the image and kernel of T . Prove that if V is finite-dimensional then V is the direct sum of $\text{im } T$ and $\text{ker } T$.
(b) Give a counterexample to the above assertion when V is infinite dimensional.
(c) Give an example where $\dim V = 4$ and $\dim(\text{im } T + \text{ker } T) < 4$.
- (2) Show that every complex $n \times n$ matrix is similar to its transpose. It helps to first consider the case of a single Jordan block.

- (3) Consider the matrix

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

- (a) Find the eigenvalues of A .
- (b) Find an orthogonal basis of eigenvectors of A .

Advanced Calculus

- (4) Let f be a continuous function on $[a, b]$. If

$$\int_a^b f(x)g(x) dx = 0$$

for all continuous functions g on $[a, b]$, show that f is identically zero on $[a, b]$.

- (5) Find A and B such that

$$\lim_{x \rightarrow 0} \frac{e^{x^2} + A \cos Bx}{x^4}$$

exists, and calculate the resulting limit.

- (6) Find the positively oriented (counterclockwise) simple closed curve C which maximizes the value of the line integral

$$\int_C (x^2y + y^3)dx + x dy.$$

- (7) For which integers k is the function

$$f(x) = \begin{cases} x^k & \text{if } x = 1/n, n \text{ a positive integer,} \\ 0 & \text{otherwise.} \end{cases}$$

continuous at $x = 0$? For which k is it differentiable at $x = 0$?